

Essays in Asset Allocation

A dissertation presented by
Thomas Leirvik

Supervised by
Prof. Fabio Trojani

Submitted to the
Faculty of Economics
Università della Svizzera italiana

for the degree of
Ph.D. in Quantitative Methods

February 2014

Doctoral Thesis

Thomas Leirvik

March 12, 2014

Chapter 1

Introduction to the Thesis

Portfolio choice and asset allocation play a central role in financial economics, and has been extensively studied the last forty years, beginning with the seminal papers of Merton (1969) and Merton (1971). It is not difficult to see why these topics attract the attention that they do; as a field of study it is both highly practical and amenable to the application of sophisticated mathematics. Ultimately, the goal of academic work in asset allocation is the conversion of the time series of observable returns and other variables of interest into a single number. Given the preferences and horizon of the investor, what fraction of wealth should the investor put in stocks and bonds?

A large body of empirical work has accumulated, documenting excess stock return predictability, for example, Campbell and Shiller (1988) and Ang and Bekaert (2007). Among the most popular predictors are the nominal interest rate and the dividend yield, and, more recently, the ratio between labor income and consumption. Vast amounts of papers investigate the motivation behind the predictability. The implications for portfolio choice problems have been investigated thoroughly, for example, by Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000), Xia (2001), and Avramov (2004). The majority of the literature on asset allocation under predictability analyzes a portfolio of cash and stocks only, and, hence assumes that the interest rate is constant. The study by Brennan et al. (1997) is an exception, and the authors consider a portfolio consisting of cash, bonds, and stocks. The returns on the stocks are predictable and the authors find the optimal allocation numerically. The papers by Korn and Kraft (2001), and Munk and Sørensen (2010) analyzed optimal portfolios with a stochastic interest rate. Much is known about optimal trading policies when the portfolio consists of one risky asset and one risk-less asset. Considerably less is known about the interaction between trading strategies for bonds and stocks when the interest rate is stochastic, expected returns are

time-varying and correlation between the asset classes is nonzero. The first chapter of this thesis is related to the paper by Brennan et al. (1997). However, unlike Brennan et al., the focus of this chapter is on an analytical solution to the investment problem. The main contribution of this chapter of the thesis will be to combine the features of stochastic real rates and a time-varying risk premium for the stock into one paper.

Another strand of literature investigates a feature often assumed constant in asset pricing models: market liquidity. The literature on asset pricing often assumes an ideal world where market participants in frictionless financial markets set security prices. In contrast to the usual assumptions on asset pricing models, Amihud and Mendelson (1989) showed that two assets exposed to the same risk factors with the same maturity and paying the same dividends had different rate of returns.

Amihud and Mendelson (1989) found that the reason for the difference in returns was the liquidity of the two assets, where the asset with higher liquidity had a lower rate of return than the asset with less liquidity. They concluded that an asset with higher liquidity offers the option to sell easily at a later point in time and, hence, sells at a premium compared to an asset with lower liquidity. In the second chapter of this thesis, the optimal allocation of wealth to a bond, or bond index, is considered when the price of the bond is affected by market liquidity.

While interest rates are recognized as the main driver of bond prices, several recent studies have pointed to the role of liquidity, as well as how the impact of time-varying market liquidity affects the prices of bonds. These include Amihud and Mendelson (1991), Duffie and Singleton (1999), Chen et al. (2007), Fontaine and Garcia (2012), Bao et al. (2011), and Amihud et al. (2013). These studies all argue that liquidity is an important factor in determining bond prices. In particular, Bao et al. (2011) established a strong link between the liquidity and the price of bonds. The authors found that changes in market-level liquidity explained a substantial part of time variations in the yield spreads of high-rated bonds, overshadowing the credit risk component. Although a precise definition of liquidity and its quantification will depend on the specific model considered, two properties are common. First, liquidity arises from market frictions, such as costs, constraints on trading, and capital flows; second, its impact on the market is transitory. These two properties indicate that a stochastic process describing liquidity should capture a time variation around a long-run mean (see Chordia et al. (2000), Hasbrouck and Seppi (2001) and Acharya and Pedersen (2005)).

It is shown that the bid-ask spread measure introduced by Corwin and Schultz (2012) possesses the desired properties of mean reversion in the market liquidity of bonds. The

bid–ask spread estimator is computed for approximately 950 different investment grade corporate bonds traded in the period of 2004–2012. The spread estimator is found to be highly time varying and mean-reverting with a peak during the financial crisis of 2008 and significant increase in volatility in the sovereign debt crisis in 2011–2012. The bid–ask spread can then be represented as a mean–reverting continuous time process and the corresponding parameters can be estimated. This is applied in a portfolio choice problem, where an investor optimally allocates wealth between a risk–free asset and a bond.

It was found that the trading policy can be decomposed into three distinct components. One component is usually denoted as the mean-variance portfolio, as it reflects the investments if there is no intertemporal hedging. The other two components describe the demand for hedging interest rate risk and liquidity risk, respectively. Moreover, the mean variance component and the hedging term for liquidity risk both depend on the spot level of the liquidity. The size of the liquidity hedge demand for liquidity risk for finite horizon investors may be as much as 50 percent larger than the demand for hedging interest rate risk.

Contents

1	Introduction to the Thesis	2
2	Life-Cycle Asset Allocation Under Stochastic Interest Rates and Stock Return Predictability	9
2.1	Introduction	10
2.2	Description of the Model	14
2.2.1	Financial Assets	15
2.2.2	Derivation of Wealth Dynamics	19
2.2.3	Investor Preferences	20
2.2.4	Hamilton-Jacobi-Bellman Equation	21
2.2.5	Optimal Strategies	22
2.2.6	Dynamic Nonmyopic Asset Allocation	24
2.2.7	Portfolios	32
2.2.8	Bond–Stock Ratio (BSR)	41
2.3	Conclusion	43
3	Optimal Portfolio Choice under Time–Varying Interest Rates and Bid–Ask Spreads	45
3.1	Introduction	46
3.2	Data Description and Summary	49
3.3	The Bid-Ask Spread	52
3.3.1	The Bid-Ask Spread Estimator	54
3.4	Description of the Model	57
3.4.1	Financial Assets	57
3.4.2	Investor Preferences	59
3.4.3	Optimal Portfolio Choice	60
3.4.4	Portfolio Decomposition	63

3.4.5	Allocation with Respect to Liquidity	66
3.4.6	Conclusion	67
Appendices		72
A Proofs for and Complimentary Results for Chapter 1		73
A.1	Proof of Proposition 2.2.3:	73
A.2	A Verification Theorem	79
A.2.1	Solution of the Control Problem	86
A.2.2	Solution of the functions $h_i(t)$	94
B Proofs and Complimentary Results for Chapter 2		98
B.1	Results and Proofs	98
B.1.1	Approximations of the Functions $g_i(t)$	109
4 Conclusion and Future Work		112
4.1	Conclusion of the Thesis	112

List of Figures

2.1	Three Predictors of Stock Market Returns	27
2.2	Optimal Asset Allocation over Time and Risk Aversion	33
2.3	The Bond Stock Mix in Over- and Undervalued Stock Markets	35
2.4	Initial Asset Allocation over the Length of the Investment Period	36
2.5	Decomposition of Bond and Stock Portfolios	38
2.6	The Bond-to-Stock Ratio in Four Different Economic Regimes	43
3.1	Time-Series Illustration of the Interest Rate	51
3.2	A Time-Series Illustration of the Bid-Ask Spread Estimator	55
3.3	A Scatterplot of the Interest Rates and Bid-Ask Spreads	57
3.4	Optimal Allocation in Bonds over Time and Risk Aversion	64
3.5	The Bond Mean-Variance Portfolio	69
3.6	The Bond Hedge Portfolios	70
3.7	Optimal Portfolio Allocation with respect to Bid-Ask Spreads	71

List of Tables

2.1	Parameter estimates	25
2.2	Asset Allocation with respect to the Labor-Income to Consumption Ratio in an Undervalued Stock Market	30
2.3	Asset Allocation with respect to the Labor-Income to Consumption Ratio in an Overvalued Stock Market	31
2.4	Optimal Asset Allocation with respect to the Dividend-Price ratio in an Undervalued Stock Market	40
2.5	Optimal Asset Allocation with respect to the Dividend-Price ratio in an Overvalued Stock Market	41

Chapter 2

Life-Cycle Asset Allocation Under Stochastic Interest Rates and Stock Return Predictability

Abstract

I analyze the optimal allocation of wealth to cash, bonds, and stocks when the interest rate is stochastic and the stock index has a time-varying mean. I find that, under certain economic conditions, the investor may optimally increase investments in stocks and bonds at the same time, which is due to the dynamic trading policies and the correlation between the asset classes. I also find that in different economic regimes, short-term investors have very different investment policies than long-term investors. Thus, dynamic asset allocation with nonzero bond-stock correlation helps explain why, during extreme market conditions such as the recent financial crisis, some investors sold all types of assets short, whereas other investors considered it an unprecedented buying opportunity.

2.1 Introduction

In this paper, I analyze the optimal dynamic allocation of wealth to three distinct asset classes for an investor with finite horizon when the interest rate and risk premium for stocks are stochastic. The solution to the stock allocation problem under a stochastic mean has been explored in the previous literature, see for example Kim and Omberg (1996). A portfolio where stochastic interest rates are considered is analyzed by Korn and Kraft (2001), Ringer and Tehranchi (2006) and further examined with Epstein-Zin preferences

in Kraft et al. (2013). The main contribution of this paper is therefore to combine the two features of stochastic real rates and a stochastic mean of the stock return into a single paper. I consider two state variables that predict future returns on the stock index, the labor income to consumption ratio (LICR), and the dividend price ratio (DPR). Recently, the LICR has been identified as an economic and statistical significant predictor of stock market returns and affects the way investors optimally allocate wealth over their lifetime, see for example Santos and Veronesi (2006). Benzoni et al. (2007) show that when the LICR is accounted for in a dynamic portfolio consisting of cash and stocks, the life-cycle portfolio is more consistent with empirical observations. The DPR is another variable recognized as a significant predictor of stock market returns, see for example Campbell and Shiller (1988), Xia (2001), Sangvinatsos and Wachter (2005), and Ang and Bekaert (2007).

In this paper, I examine how predictable variables, such as the real rates of interest, LICR, or the DPR, affect the optimal allocation of wealth to three distinct asset classes: cash, bonds, and stocks. There are three main findings of this paper. First, I find that the allocation of wealth is smoothed between bonds and stocks in ways not necessarily accounted for in a static portfolio setting. The investor may increase investments in bonds and stocks at the same time. Second, I find that the LICR has a greater impact on the optimal asset allocation than the DPR does. Third, I find that, due to predictable stock returns, the ratio of investments in bonds over stocks will, in certain economic conditions, decrease towards the investment horizon, contradicting results on asset allocation when stock returns are i.i.d, see Canner et al. (1997). These results indicate that dynamical asset allocation provides economic insight to the allocation of wealth for an investor with a finite horizon, for example, retirement.

Stock return predictability has received enormous attention over the last two decades, for example Campbell and Shiller (1988) and Ang and Bekaert (2007)¹. Stock return predictability is, however, not without controversy, as the papers by Welch and Goyal (2008), Campbell and Thompson (2008) and Cochrane (2008) show. The majority of the literature on asset allocation under predictability analyzes a portfolio of cash and stocks only. Brennan et al. (1997) are an exception, and the authors consider a portfolio consisting of cash, bonds, and stocks. The returns on the stocks are predictable and the authors find the optimal allocation numerically. This paper is closely related to the paper

¹The implications for portfolio choice problems have been investigated thoroughly: for example, Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000), Xia (2001), and Avramov (2004). Barberis solves a portfolio problem when the stock returns are predictable. Xia also solves a portfolio problem when returns are predictable, but in contrast to Barberis, Xia assumes that the investor is uncertain about the distribution of the predictable variable.

by Brennan et al. (1997), but unlike Brennan et al., I focus on an analytical solution. I find analytical expressions for the optimal value function and the corresponding optimal trading strategies. Wachter (2010) reviews the portfolio choice literature with a special emphasis on stock return predictability. Wachter considers a general setting with cash, bonds, and stocks for a preliminary analysis but the analytical results are given for a cash and stock portfolio.

Dynamic portfolio allocation problems have been studied for several decades, at least since Merton (1969).² In the standard continuous-time portfolio theory, the investor maximizes expected utility by means of continuous rebalancing of the portfolio during the investment period. The purpose of rebalancing is to adapt to shifts in wealth, interest rates, the shortening of the investor's horizon as time passes, and various other factors that affect the investor's expectations of the evolution of the investments. Much is known about the optimal trading policies when the portfolio consists of one risky asset and one risk-less asset. Kim and Omberg (1996) and Chacko and Viceira (2005) show that under certain properties of stock returns, it is possible to find analytical solutions to the two-asset portfolio problem. Considerably less is known about the interaction between the trading strategies for bonds and stocks when the interest rate is stochastic; expected returns are time-varying and correlation between the asset classes is nonzero.

In this paper, I investigate this unexplored relationship. With a power utility, the investor has a finite horizon and maximizes expected utility of terminal wealth. The risk-free rate and the stochastic risk premium on the stock are both assumed to follow the Vasicek process, see Vasicek (1977), which is a stationary mean-reverting Markovian process with normally distributed increments. The analytical solution for the investment policy is derived, as is the optimal value function. This function is obtained by means of solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation and verifying that all conditions for optimality hold. The trading policy consists of expressions for the optimal share of wealth that should be allocated to bonds and stocks where the residual is invested in the risk-less asset, also referred to as the bank account or cash. There is no consumption from or labor income added to the portfolio, thus all gains and losses are due to the trading policy. This means that the portfolio is self-financing.

The solutions provide answers to several questions regarding dynamic nonmyopic asset allocation. First, how does an investor finance investments in stocks and bonds? Second,

²The framework of Merton (1969) has been extended in several ways: Aase (1984) and Liu et al. (2003) include a discontinuous term, Kim and Omberg (1996) and Liu (2007) allowed the risk premium to be stochastic, and Chacko and Viceira (2005) relaxed the assumptions on investor preferences and considered stock returns with stochastic volatility, and Viceira (2001) considered a portfolio choice problem when labor income is accounted for.

how do investments in bonds and stocks change relative to each other and to risk aversion over time? Third, when is an investor long or short concerning any of the risky assets? Fourth, how does the correlation between the asset classes affect the investment policy? Fifth, how does the correlation between the asset classes and the stochastic risk premium affect the investment policy? And finally, when does an investor hold more or less of the risky assets than the myopic portfolio? I assume that the variables that predict expected returns on these assets are the instantaneous interest rate, the yield on the zero-coupon bond, and a vector, X , of stochastic variables affecting the expected returns of the stock.

In the analytical solution of the portfolio problem, I will make the simplifying assumption that X consists of only one variable, either LICR or the DPR on the equity portfolio. This means that I study two separate cases. First, I study the DPR as predictor, and, second, I investigate the LICR as a predictor of future stock market returns. Both these variables have a significant negative correlation coefficient with stock returns. The estimated correlation coefficient between the stock market and the DPR is -0.77 and -0.80 between the stock market and the LICR variable, respectively. This means, that in the solution of the asset allocation problem, there will be two specific variables that predict expected asset returns: the interest rate, denoted r , and the predictor X . I find that LICR has more impact on the optimal asset allocation than the DPR does. This is partly due to the stronger correlation and partly due to the lower speed of mean-reversion, which makes the signal from the predictor more reliable and persistent. I find that the nonmyopic portfolio is never identical to the myopic portfolio, except at the investment horizon, as there will be no need to hedge future shifts in the opportunity set. At the investment horizon, the investor is assumed to liquidate the risky positions in the portfolio, and keep the total wealth in a risk-less asset; for example, as a bank deposit. At all other times, the investor will hedge against both interest rate risk and changes in the opportunity set due to the predictable variable, X . Because the relationship between stocks and bonds has important implications for asset allocation and risk diversification, I analyze in detail the implications of the stock-bond correlation and bond-predictor correlation on the optimal portfolio policies. Over the time period 1959–2011, a five-year rolling correlation between the yield on the 10-Year constant maturity bond (GS10) and the DPR gives an average correlation of 0.20, with a standard deviation of 0.57. Over the same time period, the average five year rolling correlation between the S&P 500 and the GS10 is 0.02 with a standard deviation of 0.62. I find that there are several settings in which the investments are different than what is found in the static portfolio theory; for example, that the investor optimally increases investments in both stock and bonds at the same time. Further,

to find analytical solutions to the investment problem, I propose a novel method to reduce the dimension of the corresponding HJB equation. The method resembles the method of Zariphopoulou (1999), but is slightly more complicated. Moreover, as Korn and Kraft (2004) emphasized, verification of the conjectured solution is often skipped because it is mathematically demanding. Indeed, Kim and Omberg (1996) and Liu (2007) did not provide any verification conditions although the former examined the finiteness of the value function carefully. I prove that the proposed valued function and corresponding trading policies are in fact the optimal with a verification theorem.

2.2 Description of the Model

Throughout the paper, I consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)$ where $\mathcal{F}_t, t \in [0, T]$ is the filtration generated by the \mathbb{R}^3 -valued Brownian motion $Z = Z_t$. The investor trades three assets: a risk-free asset, a risky bond, and a risky stock index in a frictionless continuous-time market. I do not impose any short sale constraints, so the fractions of wealth invested in the bond and stock, $\pi_B(t)$ and $\pi_S(t)$, respectively, may very well be negative. The remainder of wealth is held in cash and equals $1 - \pi_B(t) - \pi_S(t)$, thus the investments in all three assets sum to one. The only constraint is that the value of the portfolio must be nonnegative at all times, $w_t \geq 0$. Following Merton (1971), the investor is assumed to maximize the expected value of a von-Neumann-Morgenstern utility function defined over wealth at the horizon, $w(T)$, by choosing an optimal portfolio strategy, π_t , during the entire investment period, $t \in [0, T]$. π_t is a two-dimensional vector which consists of the trading policies for the bond and the stock index, $\pi_B(t)$ and $\pi_S(t)$. A trading policy reflects the share of wealth the investor allocates to the different asset classes. In the remainder of the paper, I will omit, for brevity, parentheses with the time-variable in the trading policies, so π_B and π_S should be understood as $\pi_B(t)$ and $\pi_S(t)$, respectively.

2.2.1 Financial Assets

I assume that, under the measure P , the short-term interest rate, r_t , follows the Vasicek (1977) model

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r dZ_r(t). \quad (2.1)$$

The coefficients $\kappa_r, \bar{r}, \sigma_r$ are assumed positive and constant and Z_r is a standard Brownian motion. The price of a zero-coupon bond paying one unit of account at maturity T_1 is

then given by:³

$$B(t, T_1, r_t) = \exp(-A_1(t, T_1) - A_2(t, T_1)r_t) \quad (2.2)$$

where $\tau = T_1 - t$ is the time left to maturity and

$$A_1(t, T_1) = R_\infty[\tau - A_2(t, T_1)] + \frac{\sigma_r^2}{4\kappa_r}A_2(t, T_1)^2.$$

R_∞ is given by

$$R_\infty = \bar{r} + \frac{\sigma_r \lambda_r}{\kappa_r} - \frac{\sigma_r^2}{2\kappa_r^2}$$

and is the limit of the yield of a zero-coupon bond as maturity goes to infinity. The market price of interest rate risk is denoted as λ_r . I will follow the common assumption that the market price of risk is constant when applying the Vasicek process for the interest rate in a portfolio optimization problem, see for example, Sørensen (1999), Korn and Kraft (2001), Kraft (2003), Puhle (2007), and Munk and Sørensen (2010).⁴ The correlation between the interest rate and the bond price is $\rho_{rB} = -1$. The function $A_2(\cdot)$ is given by

$$A_2(t, T_1) = \frac{1}{\kappa_r}(1 - e^{-\kappa_r \tau}). \quad (2.3)$$

Through a standard Girsanov transformation and applying Ito's Lemma on the bond price given in Equation (2.2), the dynamical bond price is given by the following result.

Proposition 2.2.1 *Under the risk-neutral probability measure Q , the bond price dynamics are given by*

$$dB_t = r_t B(t)dt + \sigma_r A_2(t, T_1) B(t) dW_r(t). \quad (2.4)$$

where $W_r(t)$ is a Brownian motion under Q .

Proof: Let

$$dW_i = \lambda dt + dZ_i. \quad (2.5)$$

By Girsanov's theorem,⁵ there exists a process λ such that W given in Equation (2.5) is the increment of a Brownian motion under the risk-neutral probability measure Q . Substituting Equation (2.5) into the expression of the interest rate given in Equation (2.1), I get the

³The price of a bond at time t with maturity T_1 , can be written as $B(t, T_1, r) = \mathbb{E}_Q^t[\exp(-\int_t^{T_1} \tilde{r}_s ds)]$, where \tilde{r} is the interest rate under the risk-neutral probability measure Q and the superscript t denote that the expectation is taken conditional on time t . See, for example Bjørk (2009).

⁴The market price of risk can be represented by a deterministic function along the theory in Bjørk (2009), however, it will not add any significant insight in this setting.

⁵see for example Øksendal (2003)

following dynamical expression of the risk-neutral interest rate:

$$dr_t = (\kappa_r(\bar{r} - r_t) + \sigma_r \lambda_r)dt - \sigma_r dW_r(t).$$

Further, the bond price in Equation (2.2), can be written as

$$B(t, T) = \exp(Y(t, T))$$

where

$$Y(t, T) = \int_t^T \frac{\sigma_r^2}{2} A_2^2(s, T) ds - \int_t^T \kappa_r \bar{r} A_2(s, T) ds - \int_t^T \sigma_r \lambda_r A_2(s, T) ds - A_2(t, T) r(t)$$

Thus, the Ito formula gives, when time-subscripts are suppressed,

$$dB = B dY + \frac{1}{2} B (dY)^2 \quad (2.6)$$

And it remains to find dY and $(dY)^2$.

$$dY = -\frac{\sigma_r^2}{2} A_2^2 dt + \kappa_r \bar{r} A_2 dt + \sigma_r \lambda_r A_2 dt - dr A_2 - r dA_2 - dr dA_2$$

which, after inserting the Q -dynamics of the interest rate ($dr = (\kappa_r(\bar{r} - r) + \sigma_r \lambda_r)dt - \sigma_r dW_r$), gives

$$dY = -\frac{\sigma_r^2}{2} A_2^2 dt + r dt + \sigma_r A_2 dW_r$$

since $\kappa_r A_2 = 1 - \exp(-\kappa_r \tau)$, where $\tau = T - t$. The expression of $(dY)^2$ is found by squaring dY and applying the fact that $dW dt = (dZ + \lambda dt)dt = 0$ and $dW^2 = dt$:

$$(dY)^2 = \sigma_r^2 A_2^2 dt$$

Hence, inserting the expressions of dY and $(dY)^2$ into dB in Equation (2.6), gives

$$dB = B[r A_2 dt + \sigma_r A_2] dW_r \quad (2.7)$$

which completes the proof. \square

To find the optimal trading policies, it is necessary to have the dynamical bond price under the measure P , which follows directly from Proposition 2.2.1:

Lemma 2.2.1 *The bond price dynamics under P are given by*

$$\begin{aligned} dB_t &= (r_t + \sigma_r \lambda_r A_2(t, T_1))B(t)dt + \sigma_r A_2(t, T_1)B(t)dZ_r(t). \\ &= B(t)[(r_t + \sigma_B(t)\lambda_r)dt + \sigma_B(t)dZ_r(t)]. \end{aligned} \quad (2.8)$$

Proof of Lemma 2.2.1:

Since $dW_r = dZ_r + \lambda_r dt$, the P -dynamics of the bond price are found by substituting the expression of dW_r into Equation (2.4), and I get

$$dB = rBdt + \sigma_r A_2 B dW_r = (rB + \sigma_r \lambda_r A_2 B)dt + \sigma_r A_2 B dZ_r$$

which completes the proof. \square

In Equation (2.8) the term $-\sigma_r A_2(t, T_1)$ is written compactly as $\sigma_B(t)$. In addition to the bond, the agents can invest in a stock, representing a stock market index, with price dynamics

$$\frac{dS_t}{S_t} = (\mu + r_t + \beta_S(X_t - \bar{X}))dt + \sigma_S(\rho_{SB}dZ_r + \sqrt{1 - \rho_{SB}^2}dZ_S(t)) \quad (2.9)$$

where $Z_S = Z_S(t)$ is a standard Brownian motion independent of Z_r , μ is the constant expected excess return not due to the predictor X . The coefficients β_S and σ_S are assumed constant. β_S describes how much the drift of the stock price, or stock index, is affected by the predictable variable, see for example, Xia (2001). σ_S describes the volatility of the stock price. The coefficient ρ_{SB} describes the constant correlation between the stock and the bond. The prices of the risky assets can be written as the vector $P_t = (B_t, S_t)^\top$ and hence the dynamics of P_t can be written as

$$dP_t = \text{diag}(P_t)[(r_t \cdot \mathbf{1} + \Sigma_t \Lambda)dt + \Sigma_t dZ] \quad (2.10)$$

where $dZ = (dZ_r, dZ_S)^\top$, $\Lambda = \lambda + \hat{\mu}$, where

$$\hat{\mu} = (0, \frac{\beta_S(X - \bar{X})}{\sigma_S \sqrt{1 - \rho_{SB}^2}})^\top.$$

This gives the following expression for Σ_t :

$$\Sigma_t = \begin{bmatrix} \sigma_B(t) & 0 \\ \sigma_S \rho_{SB} & \sigma_S \sqrt{1 - \rho_{SB}^2} \end{bmatrix} \quad (2.11)$$

and the market price of risk is the vector $\lambda = (\lambda_r, \lambda_S)^\top$, where λ_S is given by

$$\lambda_S = \frac{1}{\sqrt{1 - \rho_{SB}^2}} \left(\frac{\mu}{\sigma_S} - \rho_{SB} \lambda_r \right)$$

I further assume that the predictor, X , is described by the following Vasicek process:

$$dX_t = \kappa_X (\bar{X} - X_t) dt + \sigma_X (\rho_{XP}^\top dZ + \sqrt{1 - \|\rho_{XP}\|^2} dZ_X(t)) \quad (2.12)$$

where X_t is the level of the predictor and \bar{X} is its long-run mean. $Z_X = Z_X(t)$ is a standard Brownian motion independent of Z_r and Z_S . κ_X is the speed of mean reversion, whereas σ_X is the volatility of the predictor. The correlation between the predictor and the price process given in Equation (2.10), is denoted as ρ_{XP} , and given as the vector $(\rho_{XB}, \hat{\rho}_{XS})^\top$, where ρ_{XB} is the correlation between the predictor and the bond and $\hat{\rho}_{XS} = \frac{\rho_{XS} - \rho_{SB}\rho_{XB}}{\sqrt{1 - \rho_{SB}^2}}$, where ρ_{XS} is the correlation between the stock and predictor.

2.2.2 Derivation of Wealth Dynamics

Recall that if the investor allocates an admissible fraction π_S of wealth in the stock, the fraction π_B in the bond and hence $1 - \pi_B - \pi_S$ in the risk-less bank-account, then the total wealth is given by

$$w_t = \frac{\pi_S w_t}{S} S + \frac{\pi_B w_t}{B} B + \frac{(1 - \pi_B - \pi_S) w_t}{R} R$$

where R is the amount in the risk-less account and evolving according to $dR = r_t R dt$. The self-financing condition leads to the dynamical expression

$$dw_t = \frac{\pi_S w_t}{S} dS + \frac{\pi_B w_t}{B} dB + \frac{(1 - \pi_B - \pi_S) w_t}{R} dR$$

Thus, substituting the expression of the risk-less bank account dR , the bond price given in Equation (2.8), combined with the dynamical expression of the stock price given in Equation (2.9), gives the following result:

Proposition 2.2.2 *The P-dynamics of the portfolio, or wealth, w is given by*

$$dw = (r_t \cdot \mathbf{1} + \Sigma_t \Lambda \pi_t^\top) w dt + w \pi_t^\top \Sigma_t dZ \quad (2.13)$$

Proof:

The P-dynamics of the portfolio w_t , when the expressions of the cash and bond and stock

prices are collected in Equation (2.10)

$$\begin{aligned} dw_t = & \frac{\pi_S w_t}{S} S(\hat{\mu}_t dt + \sigma_S(\rho_{SB} dZ_r + \sqrt{1 - \rho_{SB}^2} dZ_S(t))) + \frac{\pi_B w_t}{B} ((r + \sigma_r \lambda_r A_2) B dt \\ & + \sigma_r A_2 B dZ_r) + \frac{(1 - \pi_S - \pi_B) w_t}{R} r R dt \end{aligned}$$

where $\hat{\mu}_t = (\mu + r_t + \beta_S(X_t - \bar{X}))$. So, collecting the expressions with dt , dZ_r , and dZ_S , I get

$$dw_t = \left(r + \pi_S(\hat{\mu} - r) + \pi_B \sigma_r \lambda A_2 \right) w_t dt + \pi_B w_t \sigma_r A_2 dZ_r + \pi_S w_t \sigma_S dZ_S$$

Thus, stacking the result in terms of a vector, I get the following

$$dw_t = (r_t \cdot \mathbf{1} + \Sigma_t \Lambda \pi_t^\top) w_t dt + w_t \pi_t^\top \Sigma_t dZ \quad (2.14)$$

where $\sigma_B(t) = \sigma_r A_2(t, T_1)$ and

$$\Sigma_t = \begin{bmatrix} \sigma_B(t) & 0 \\ \sigma_S \rho_{SB} & \sigma_S \sqrt{1 - \rho_{SB}^2} \end{bmatrix}. \quad (2.15)$$

Which completes the proof. \square

2.2.3 Investor Preferences

I assume that the investor has isoelastic utility so that the utility of the bequest function is defined as

$$U(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad (2.16)$$

where $\gamma > 1$ is the risk aversion and the wealth is assumed to be nonnegative. Given the opportunity to invest in the risk-less asset and the two risky assets, the investor starts with a positive wealth, $w_0 > 0$, and chooses at each time, t , to invest a fraction π_t of the wealth in the assets and so seeks to maximize expected utility of terminal wealth w_T ,⁶

$$V(t, w, r, X) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}_t[U(w_T)], \quad \text{for } t \in [0, T] \quad (2.17)$$

⁶ A constant discount term inside the expectation in Equation (2.17) is omitted as there is no intermediate consumption and as the interest rate r takes care of discounting final wealth, see Ingersoll (1987), Korn and Kraft (2001), and Puhle (2007).

where \mathcal{A} is the set of all admissible controls $\pi_t = (\pi_B(t), \pi_S(t))^\top$ and the wealth process satisfies the self-financing condition

$$dw = (r_t \cdot \mathbf{1} + \Sigma_t \Lambda^\top \pi_t^\top) w dt + w \pi_t^\top \Sigma_t dZ \quad (2.18)$$

The drift and diffusion term of the wealth consists of the interest rate plus the drift and diffusion coefficients of the price process given in Equation (2.10).

2.2.4 Hamilton-Jacobi-Bellman Equation

The principle of optimal stochastic control leads to the HJB equation, which corresponds to the processes described in Equations (2.1), (2.12), and (2.18),

$$\begin{aligned} \mathcal{L}V(t, w, r, X) &= V_t + (r_t \cdot \mathbf{e}_1 + \Sigma_t \Lambda^\top \pi_t) w V_w + \frac{1}{2} w^2 \pi_t^\top \Sigma_t \Sigma_t^\top \pi_t V_{ww} + \kappa(\bar{X} - X) V_X \\ &+ \kappa_r(\bar{r} - r) V_r + \frac{\sigma_r^2}{2} V_{rr} + \frac{\sigma_X^2}{2} V_{XX} - \sigma_r \sigma_X \rho_{XB} V_{rX} \\ &+ \sigma_X \pi_t^\top \Sigma_t \rho_{XP} w V_{wX} - \sigma_r \pi_t \Sigma_t \mathbf{e}_1 w V_{wr}. \end{aligned} \quad (2.19)$$

where $\mathbf{e}_1 = (1, 0)^\top$ and subscripts on V denote a partial derivative. The first-order condition for the portfolio π_t implies that

$$\pi_t = -\frac{V_w}{w V_{ww}} (\Sigma_t^\top)^{-1} \Lambda^\top + \frac{V_{rw}}{w V_{ww}} \sigma_r (\Sigma_t^\top)^{-1} \mathbf{e}_1 - \frac{V_{Xw}}{w V_{ww}} \sigma_X (\Sigma_t^\top)^{-1} \rho_{XP} \quad (2.20)$$

The first term of Equation (2.20) represents the standard mean-variance portfolio, whereas the last two terms represent the hedge against the interest rate and the hedge against predictor risk. Because the bond is perfectly negatively correlated with the interest rate, the interest rate is hedged by a position in the bond only. To circumvent a dimensionality problem that often arises when considering the HJB equation, I propose that the indirect value function, $V(\cdot)$, is given as a product of three functions:

$$V(t, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma} f(t, X)^{\theta_1} g(t, r)^{\theta_2}. \quad (2.21)$$

The functions $f(t, X)$ and $g(t, r)$ will be determined later. I find that if the θ_i 's are given by

$$\theta_1 = \frac{\gamma}{\gamma + (1-\gamma) \rho_{XP}^\top \rho_{XP}} \quad (2.22)$$

$$\theta_2 = \gamma, \quad (2.23)$$

then some of the nonlinear terms in the HJB equation given in Equation (2.19) cancel and I show that there is an analytical solution for the differential equation that remains after canceling those terms. The factorization in Equation (A.1) is comparable to the one in Zariphopoulou (1999), and equivalent if the function $f(t, X)$ is one. This is only possible if stock returns are not predictable, and the process X given in Equation (2.12) is zero.⁷ Thus the independence assumption between the Brownian motions governing the interest rate and the predictor is important to be able to factorize in this manner.

2.2.5 Optimal Strategies

I will now state the proposed solution to the optimal value function and the corresponding trading policies.

Proposition 2.2.3 *If the stock returns are predictable and the parameter $\theta = [\theta_1, \theta_2]$ is given as in Equations (2.22) and (2.23), then the optimal value function is given by*

$$V(t, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma} \exp(h_0(t) + h_1(t)r + h_2(t)X + h_3(t)X^2) \quad (2.24)$$

where

$$V(T, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma} \quad (2.25)$$

and the optimal trading policies are

$$\begin{aligned} \pi_t^* = & \left[\frac{\lambda_r \sigma_S - (\mu + \beta_S(X - \bar{X})) \rho_{SB}}{\gamma \sigma_B(t) \sigma_S (1 - \rho_{SB}^2)} \right] - \frac{\sigma_r h_1(t)}{\sigma_B(t)} \cdot e_1 \\ & + \sigma_X \frac{\theta_1 (h_2(t) + 2X h_3(t))}{\gamma} \left[\frac{\rho_{XB} - \rho_{SB} \rho_{XS}}{\sigma_B(t) (1 - \rho_{SB}^2)} \right] \\ & \left[\frac{\rho_{XS} - \rho_{SB} \rho_{XB}}{\sigma_S (1 - \rho_{SB}^2)} \right] \end{aligned} \quad (2.26)$$

Proof is found in Section A.

where π_t^* is the vector $[\pi_B^*(t), \pi_S^*(t)]^\top$. For Equation (2.25) to hold, the functions $h_i(t)$ must all equal zero at the investment horizon, thus $h_i(T) = 0$. Given this terminal condition, the solutions to the functions $h_i(t)$ are given as a system of differential equations. For exact solutions, see Equations (A.13)-(A.18) in Section A.2.2 in the appendix. The functions $h_i(t)$ are nonzero at all other times $t \in [0, T]$ and converge monotonically to zero in time. The bond portfolio consists of three terms: the usual mean-variance portfolio,

⁷Zariphopoulou (1999) applies the factorization $V(t, w, r) = \frac{w^{1-\gamma-1}}{1-\gamma} g(t, r)^\gamma$.

the interest rate hedge demand, and the predictor hedge demand. The stock portfolio consists of two terms, the mean-variance portfolio and the predictor hedge portfolio. The mean-variance term is independent of time, whereas the predictor hedge term depends on time through $h_2(t)$ and $h_3(t)$. This implies that the predictor hedge demand converges to zero in time and eventually, at the investment horizon, $t = T$, the hedge term is zero and only the first term remains. The mean-variance term is sometimes denoted as the myopic portfolio, as it describes how the investor allocates wealth when only one time step ahead is considered. Moreover, both the bond and the stock portfolio depends on the level of the predictor, X , which implies there is an element of market timing in the optimal trading policy.

2.2.6 Dynamic Nonmyopic Asset Allocation

In this section, I will analyze a wide range of questions regarding the investment decisions for an investor who allocates wealth to cash, bonds, and stocks when stock returns are predictable. The values of the parameters applied here are estimated from quarterly data on the Standard & Poor's (S&P) 500 index, the corresponding DPR ranging from 1959 through 2011, and observations about labor income and consumption obtained from the Bureau of Economic Analysis, US Department of Commerce. Observations on the interest rate are also quarterly and ranging over the same time period. The correlation coefficient between bonds and stocks is inspired by recent literature, for example Ilmanen (2003). The correlation between bonds and stocks has been a changing sign over the years, so I will analyze the impact of both negative and positive correlation coefficients on the optimal asset allocation. I find the correlation coefficient to have economically significant impact on the portfolio allocation when stock returns are predictable. Table 3.2 represents the values of the parameters applied in the analysis in this section.

Table 2.1: Parameter estimates

This table presents the estimates of the parameters in the models describing the interest rate, the bond, the stock, and the predictable variable. The observations on labor income are obtained from the Bureau of Economic Analysis, US Department of Commerce, whereas information about the interest rate and the stock market is obtained from Robert Shiller, <http://www.econ.yale.edu/~shiller/>.

Process	Parameter description	Notation	Parameter values
Interest rate	Speed of mean reversion	κ_r	0.35
Interest rate	Long-run mean	\bar{r}	0.05
Interest rate	Volatility	σ_r	0.02
Stock returns	Excess returns	μ	0.06
Stock returns	Predictor loading DPR	β_S	3.58
Stock returns	Predictor loading LICR	β_S	0.59
Stock returns	Volatility	σ_S	0.144
LICR	Speed of mean reversion	κ_X	0.0129
LICR	Long-run mean	\bar{X}	0.75
LICR	Volatility	σ_X	0.0097
DPR	Speed of mean reversion	κ_X	0.19
DPR	Long-run mean	\bar{X}	0.04
DPR	Volatility	σ_X	0.006
Correlation	Bond-stock	ρ_{SB}	0.15
Correlation	Stock index - DPR	ρ_{SX}	-0.77
Correlation	Stock index -LICR	ρ_{SX}	-0.8
Correlation	Bond - DPR	ρ_{XB}	0.3
Correlation	Bond - LICR	ρ_{XB}	0.3

In addition to the parameter values described in table I, the asset allocation problem crucially depends on the investment horizon, T , and the maturity of the bond, T_1 . The investment horizon will be assumed to be ten years, $T = 10$, and the maturity of the bond a year after the investment horizon, $T_1 = T + 1$. In estimating the parameters of the two predictors, LICR and DPR, I apply the same continuous process as described in Equation (2.12). The impact of these predictors on the optimal asset allocation will be treated in two separate cases, thus I have used the same notation for the parameters in

both cases. I find for example that the estimated value of the long-run mean, denoted \bar{X} , is 0.75 for the LICR and 0.04 for the DPR. The spot level of the predictor itself does not have a significant impact on the portfolio strategy, even though the current level is included directly in the policies. The important matter is the deviation of the spot from the long-run mean. The level of the predictor will be represented through the loading of the predictor on the asset prices, denoted β_S , as a predictor with a low spot rate, say the DPR, must have a large loading to influence the returns. On the other hand, the LICR, which is found to have a spot rate around 0.8, has a smaller loading to represent stock returns. Xia (2001) found that $\beta_S = 3.85$ when the DPR predict stock returns. Figure 2.1 shows a time series plot of quarterly observations on the interest rate, the LICR, and the DPR.

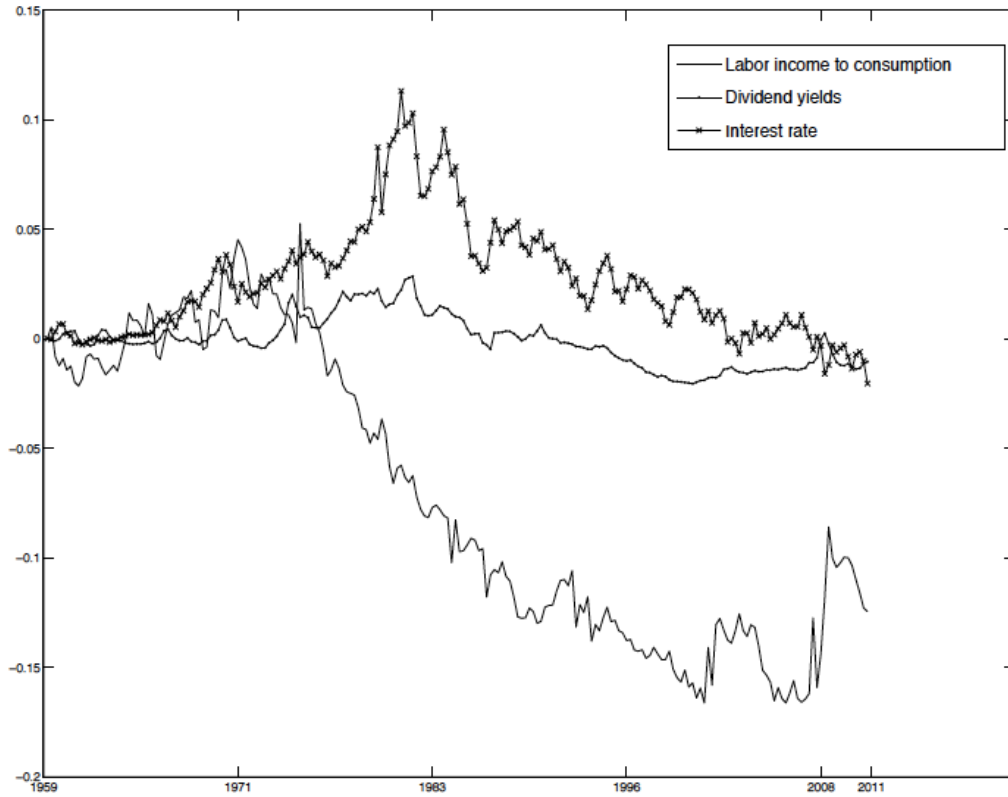


Figure 2.1: The figure illustrates the three variables predicting stock market returns; the interest rate, the LICR and the DPR. The observations are quarterly and start in 1959 and end in 2011. The data is normalized in order to be represented in the same figure. The figure indicates that DPR is by far the variable with the lowest volatility, something also found in the standard deviation of the time series and the estimated value of the volatility.

Tables

Tables 2.2–2.5 report the optimal percentage invested in bonds and stocks at the start of the investment period, $t = 0$, with respect to the correlation between bonds and stocks and length of the investment period. The length of the investment period is in years and equals T , and I have given the optimal asset allocation when $T = 0$, $T = 2$, $T = 5$ and $T = 20$. The optimal portfolios are represented by (π_B^*, π_S^*) , thus the left number in the parentheses is the optimal percentage invested in the bond, whereas the number to the right is the optimal percentage of wealth invested in stocks. The correlation between bonds and stocks ranges between $\rho_1 = -0.3$ to $\rho_1 = 0.3$. The tables decompose the portfolio into three terms: the mean variance term, the interest rate hedge term, and the predictor hedge term, π_{mv} , π_{ih} , and π_{ph} , respectively. π^* is the sum of those three terms and is the optimal allocation to bonds and stocks, $\pi^* = (\pi_B^*, \pi_S^*)$. The parameters of the interest rate, the LICR, and the DPR, are found with least squares estimation. The parameter representing the loading of the predictor on stock returns, β_S , is inspired by Xia (2001). Table 3.2 gives the estimates of all parameters applied in this section. Risk aversion equals four.

Labor Income to Consumption Ratio (LICR)

The ratio labor income to consumption (LICR), has recently been identified as an economically significant variable predicting stock market returns, see for example Menzly and Veronesi (2004) and Santos and Veronesi (2006), affecting how an investor optimally invests in stocks, for example Viceira (2001). However, these papers do not consider time-varying interest rates and the possibility for the investor to invest in bonds. In the subsequent analysis, I consider the impact of LICR on the allocation of wealth to cash, bonds, and stocks, and how investments in these asset classes varies with respect to the correlation structure between them and whether the stock market is considered under- or overvalued in terms of the deviation from the long-run mean for the current level of the LICR. I apply quarterly data on labor income and consumption starting for 1959–2011, see Figure 2.1 for a time series plot. In the dataset from the Bureau of Economic Analysis, US Department of Commerce, labor income is the aggregate wage and salary disbursements plus other income minus taxes and contributions for government social insurance. Consumption is defined as everything spent on nondurable goods and services. Both labor income and consumption is for the US as a whole and the ratio between them is denoted as LICR. Consumption in this context is not the same as consumption from the financial portfolio of the investor because the investor consider the LICR as an endogenous process

that predict expected stock returns. This means that the portfolio of the investor is self-financed as I assume an investor who do not consume from the portfolio. This implies that once money is transferred to the portfolio, it will stay there accumulating returns, positive or negative, until the investment horizon when the portfolio is liquidated. I assume that LICR can be represented by the mean-reverting Vasicek process, and I estimate the parameters incorporated in this equation. The predictor, LICR in this section, is denoted by X and the long-run mean is denoted by \bar{X} . X consists of two processes: aggregate US labor income and US consumption. Thus one can write the predictor as

$$X = \frac{\text{aggregate wages}}{\text{aggregate consumption}}$$

The change in labor income is more variable than consumption, illustrated by standard deviations of 0.014 and 0.008, respectively. This means that if the current level of the predictor is above the long-run mean, it is more likely that wages increased quickly, rather than a rapid decrease in consumption. This again implies that consumption will increase to catch up with the increased wealth of consumers, and will therefore be reflected in higher expected asset prices as more goods are consumed. Intuitively, this means that the investor should be long in stocks, as earnings for the companies are expected to increase. Table 2.2 shows the optimal investments in bonds and stocks with respect to changes in the correlation between stocks and bonds when the current level of the labor income to consumption is two standard deviations above the long-run mean. The standard deviation of the LICR is found to be 0.0659, whereas the sample average is 0.787. Table 2.2 shows the optimal allocation of wealth to stocks and bonds when the current level of labor income to consumption is above the long-run mean. The investor is then optimally long in the stock market, and as the length of the investment period increases, the investor first increases the fraction of wealth in stocks until the investment period is about ten years. If the investment period is longer than ten years, the investor decreases stock investments again, which leads to a hump-shaped trading policy for stocks. There is also evidence that the investor diversifies risk in the sense that when the bond-stock correlation is negative, bond prices are expected to decrease, and hence the investor is short in bonds. When the bond-stock correlation is positive and the investment horizon is shorter than five years, the investor is long in both asset classes, but has reduced exposure to stocks compared with the negative bond-stock correlation case.

Table 2.3 shows a different setting, namely when the current level of the LICR is two standard deviations below the long-run mean. In this setting, consumption may be higher than what the labor income indicates, and hence is likely to slow down. This will be

reflected in decreasing asset prices, as earnings will be expected to decrease due to lower consumption. This predicts lower expected asset returns, and hence the investor should be short in stocks, and depending on the bond-stock correlation, be short or long in bonds.

Table 2.2: LICR and optimal asset allocation with varying bond-stock correlation when $\bar{X} < X$

This table reports the optimal asset allocation when the level of the predictor is above its long-run mean, specifically the level of the predictor is equal to $X = \bar{X} + 2s$, where s is the sample standard deviation and equal $s = 0.0659$.

T		ρ_{SB}				
		- 0.3	-0.15	0	0.15	0.3
0	π^*	(-78.2, 47.1)	(-28.6, 43.5)	(15.4, 42.2)	(60.0, 42.8)	(112.0, 45.6)
	π_{mv}	(-78.2, 47.1)	(-28.6, 43.5)	(15.4, 42.2)	(60.0, 42.8)	(112.0, 45.6)
	π_{ih}	0	0	0	0	0
	π_{ph}	0	0	0	0	0
2	π^*	(-43.5, 76.87)	(-32.5, 67.1)	(-21.0, 63.0)	(-8.4, 62.9)	(6.8, 66.7)
	π_{mv}	(-27.4, 47.1)	(-10.0, 43.5)	(5.4, 42.2)	(21.0, 42.8)	(39.2, 45.6)
	π_{ih}	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)
	π_{ph}	(18.0, 29.6)	(11.7, 23.5)	(7.8, 20.8)	(4.8, 20.1)	(1.8, 21.1)
5	π^*	(-40.5, 75.0)	(-36.6, 66.6)	(-31.9, 63.1)	(-26.5, 63.3)	(-19.7, 67.5)
	π_{mv}	(-14.7, 47.1)	(-5.4, 43.5)	(2.9, 42.2)	(11.3, 42.8)	(21.1, 45.6)
	π_{ih}	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)
	π_{ph}	(16.9, 27.9)	(11.5, 23.0)	(7.8, 20.9)	(4.9, 20.5)	(1.8, 21.9)
20	π^*	(-43.6, 65.0)	(-42.9, 59.3)	(-41.9, 57.0)	(-40.6, 57.7)	(-38.9, 61.5)
	π_{mv}	(-5.7, 47.1)	(-2.1, 43.5)	(1.2, 42.2)	(4.5, 42.8)	(8.4, 45.6)
	π_{ih}	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)
	π_{ph}	(10.9, 17.9)	(7.8, 15.7)	(5.6, 14.8)	(3.5, 14.8)	(1.3, 15.8)

Dividend price ratio (DPR)

The dividend price ratio (DPR) is one of the major variables predicting stock market returns, see for example Campbell and Shiller (1988), and is known to have a large impact on the investment decision of investors, see for example Brennan et al. (1997) and Xia (2001). In the analysis of this section, I assume that the DPR can be represented by the continuous mean-reverting Vasicek process. To estimate the parameters of this equation, I apply quarterly data on the level of the S&P 500 Index and aggregate dividends paid by the companies in this index from 1959 through 2011. I denote the DPR with the variable X and find that the sample average is 0.031, whereas the sample standard deviation equals

Table 2.3: LICR and optimal asset allocation with varying bond-stock correlation when $\bar{X} > X$.

This table reports the optimal asset allocation when the level of the predictor is above its long-run mean, specifically the level of the predictor is equal to $X = \bar{X} + 2s$, where s is the sample standard deviation and equal $s = 0.0659$.

T		ρ_{SB}				
		- 0.3	-0.15	0	0.15	0.3
0	π^*	(65.1, -22.7)	(38.2, -21.5)	(15.4, -21.4)	(-6.7, -22.2)	(-31.3, -24.2)
	π_{mv}	(65.1, -22.7)	(38.2, -21.5)	(15.4, -21.4)	(-6.7, -22.2)	(-31.3, -24.2)
	π_{ih}	0	0	0	0	0
	π_{ph}	0	0	0	0	0
2	π^*	(2.7 0.4)	(-11.8, -3.5)	(-22.5, -5.5)	(-32.9, -7.1)	(-43.8, -8.5)
	π_{mv}	(22.8, -22.7)	(13.4, -21.5)	(5.4, -21.4)	(-2.3, -22.2)	(-10.9, -24.2)
	π_{ih}	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)
	π_{ph}	(14.0, 23.1)	(9.0 18.1)	(5.9, 15.8)	(3.6, 15.1)	(1.3 15.76)
5	π^*	(-17.8, -1.8)	(-27.0 -4.5)	(-34.0, -6.08)	(-40.4, -7.2)	(-47.2, -8.4)
	π_{mv}	(12.2, -22.7)	(7.2, -21.50)	(2.9, -21.4)	(-1.3, -22.2)	(-5.9, -24.2)
	π_{ih}	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)
	π_{ph}	(12.7, 20.9)	(8.5, 17.1)	(5.8 15.4)	(3.6, 15.0)	(1.3, 15.8)
20	π^*	(-35.8, -9.7)	(-40.1, -10.2)	(-43.5, -10.7)	(-46.6, -11.6)	(-50.0, -12.9)
	π_{mv}	(4.9, -22.7)	(2.9, -21.50)	(1.2, -21.4)	(-0.5, -22.2)	(-2.3, -24.2)
	π_{ih}	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)
	π_{ph}	(7.9, 13.1)	(5.7, 11.4)	(4.0, 10.7)	(2.5, 10.6)	(1.0, 11.3)

0.011. The DPR is defined as

$$X = \frac{Dividends}{Price}$$

and when considering an index, the *price* is the level of the index and *dividends* is the total dividends paid by all companies in the index. When the current DPR is above its long-run mean, that is $X > \bar{X}$, this may indicate that the stock is underpriced and hence expected to increase in the future. Table 2.4 shows how the investor optimally allocates wealth to bonds and stocks when the stock index is underpriced. The investor is optimally long in the stocks, meaning the fraction of wealth in stocks is positive. π_S increases as the time to the investment horizon increases. If the bond-stock correlation is negative, the bond prices are expected to decrease. The investor is then optimally short in bonds, meaning that the fraction of wealth allocated to bonds, denoted as π_B , is negative. If the bond-stock correlation is positive, the investor is long in bonds as well as stocks. When the bond-stock correlation is positive, the investor reduces the fraction of wealth in stocks

compared to when the bond-stock correlation is negative, a behavior attributable to the investor's need to diversify the risk of the portfolio.

Conversely, the stock-price may be overvalued if the current DPR is lower than the long-run mean. Table 2.5 shows the optimal allocation when the stock price is overvalued and expected to decrease. The investor will then sell the stock short. The optimal fraction invested in the bond will also in this case depend on the bond-stock correlation. If positive, then the investor sells the bond short as well, as the bond is also expected to decrease. If negative, the situation is somewhat more complicated: the optimal allocation to bonds depends on the length of the investment horizon. For short horizons, up to 1-2 years, the investor is long in the bond. For longer horizons, the interest rate risk overshadows the potential to capitalize on the bond-price movement due to the DPR, and hence the investor short sells the bond.

Furthermore, if there is no correlation between bonds and stocks, there will be no interaction between investments in stocks and bonds. The fraction of wealth invested in bonds will then be independent of X . The fraction of wealth in stocks will still depend on the DPR, but there is no smoothing across the asset classes to reduce the risk of the portfolio. Table 2.5 illustrates how the optimal bond-stock portfolio is allocated when the level of the DPR, X , is two standard deviations below its long-run mean, \bar{X} . As the table shows, the investor short sells the overpriced stock, and increases the short position as the length of the investment period increases, if the correlation between bonds and stocks is positive. If the correlation is negative, then the investor decreases the short position in stocks.

2.2.7 Portfolios

The optimal policies given in Equation (2.26) depend strongly on both time and risk aversion. The hedge demand decreases as time to investment horizon approaches and investments decreases as risk aversion increases. Figure 2.2 illustrates the optimal shares invested in the bond and the stock as functions of time and risk aversion. In Figure 2.2, the predictor, X , is assumed to be above the long-run mean, and thus expected returns on the stock are high. The bond price is expected to decrease, and the investor is short the bond and long the stock. The optimal invested percentage in bonds decreases in time as the investor shifts wealth to the stock market where the expected returns are above average. The optimal percentage invested in bonds increases in risk aversion as the investor seeks less risk and thus shifts more investments to the safer asset class of bonds. With the predictor above the long-run mean, the stock portfolio decreases in time. Because the

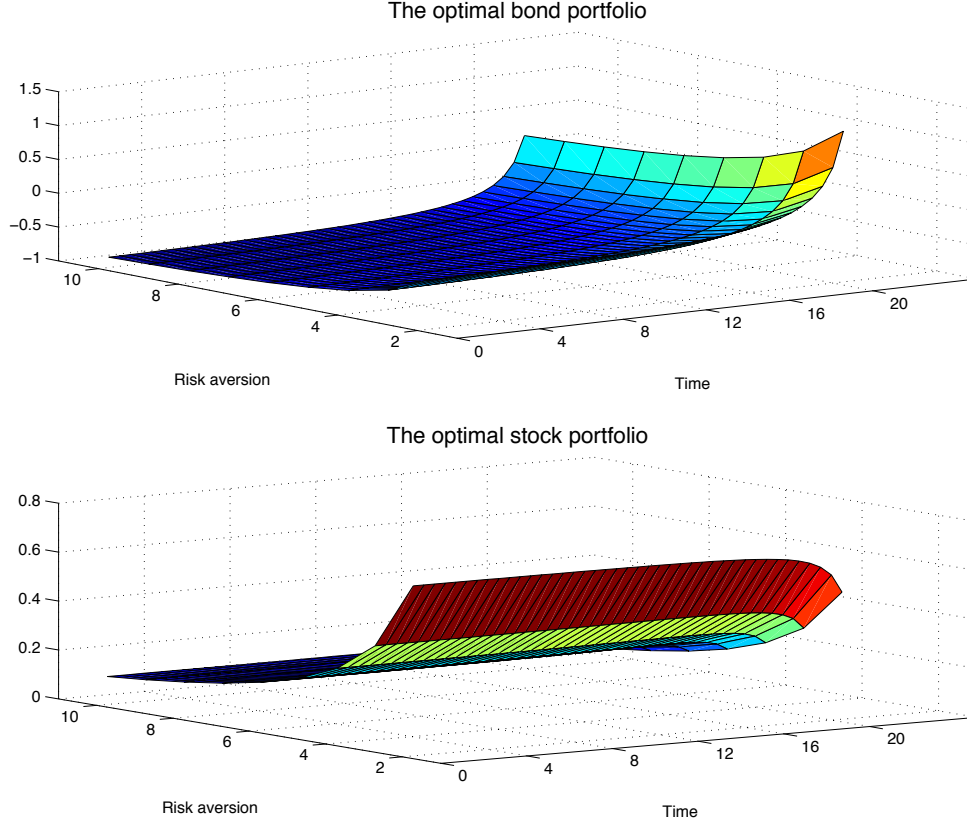


Figure 2.2: The figure illustrates the optimal percentage of wealth invested in the bond and in the stock. The bond portfolio decreases in risk aversion, and increases in time. The stock portfolio decreases in risk aversion and time. The bond-stock correlation is negative and equals $\rho_{SB} = -0.3$ and the stock price is assumed to be undervalued as the current DPR is higher than the long-run mean.

more risk averse investor seeks less risk, one expects the stock portfolio to decrease in risk aversion, and the figure shows that this is indeed the case: optimal stock investment decreases substantially in risk aversion. Figure 2.3 illustrates the asset allocation under four different economic conditions. When the stock market is undervalued, as indicated by the DPR, the optimal fraction in stocks is positive, as shown in panel *a*) and panel *b*) in Figure 2.3. In panels *a*) and *c*), the correlation between the bond and stock is positive and equals $\rho_{SB} = 0.3$, whereas in panels *b*) and *d*), it is negative and equals $\rho_{SB} = -0.3$. As panel *a*) show, the investor will be long in bonds and stocks throughout the investment period. However, the investor increases investments in stocks in the beginning of the period, and decreases investment in bonds slightly. At the end of the investment period, this reverses, and the investor decrease investments in stocks and increase investments in

bonds.. On the other hand, if the bond stock correlation is negative, then the investor start with a long position in bonds but decrease bond investments throughout the investment period, and ultimately keeps a short position in bonds, see panel b). When the stock price is overvalued, as shown in panels c) and d) in Figure 2.3, the investor keeps a negative fraction of wealth in the stock, as the price is expected to decrease. If the bond-stock correlation is positive, the optimal fraction in bonds will be negative throughout the investment period as well, but when the correlation is negative, the investor will shift the bond fraction from being negative in the beginning of the investment period to becoming positive at the end.

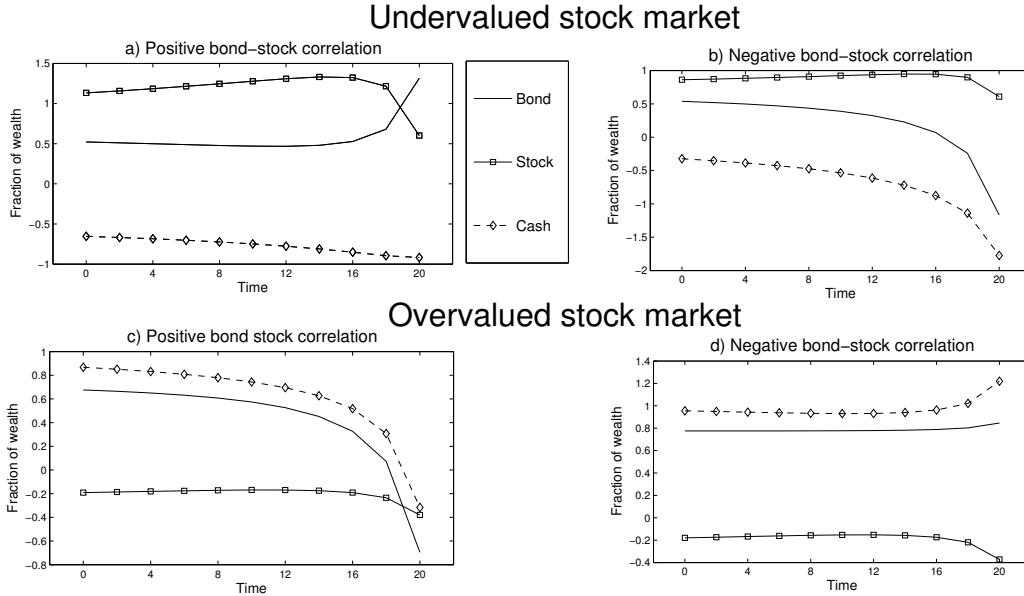


Figure 2.3: The figure illustrates the fractions of wealth invested in cash, bonds, and stocks under four different economic conditions, with respect to changes in time, when the risk aversion equals three. The correlation between the predictor and the interest rate is assumed to be $\rho_{XB} = 0.3$ and the correlation between the stock price and the predictor is $\rho_{SX} = -0.8$.

Time Horizon

An important question addressed by several authors on portfolio choice is the impact of the length of the investment period. For instance, Kim and Omberg (1996) show that the optimal allocation in stocks increases as the length of the investment period increases. However, I find that this does not necessarily hold true. The investor might decrease

investments in stocks as time to the horizon increases, which is the case when the stock price is over valued. This is also the case for investments in bonds. If the correlation is positive and has a long horizon, the investor invests more in bonds. If the correlation is negative, the investor expects lower returns on the bonds, because the spot level of the predictor is above its long-run mean, and thus decreases investments in bonds. Figure 2.4 represents how the investor allocates wealth at the beginning of the investment period, depending on the length of the investment period. The optimal stock allocation in Figure 2.4 confirms what Kim and Omberg (1996) found in their paper, that the fraction in stocks increases as the length of the investment period increases. As the figure also shows, the investor reduces the optimal fraction of wealth in bonds as the investment period becomes longer.

Optimal initial investments in bonds and stocks

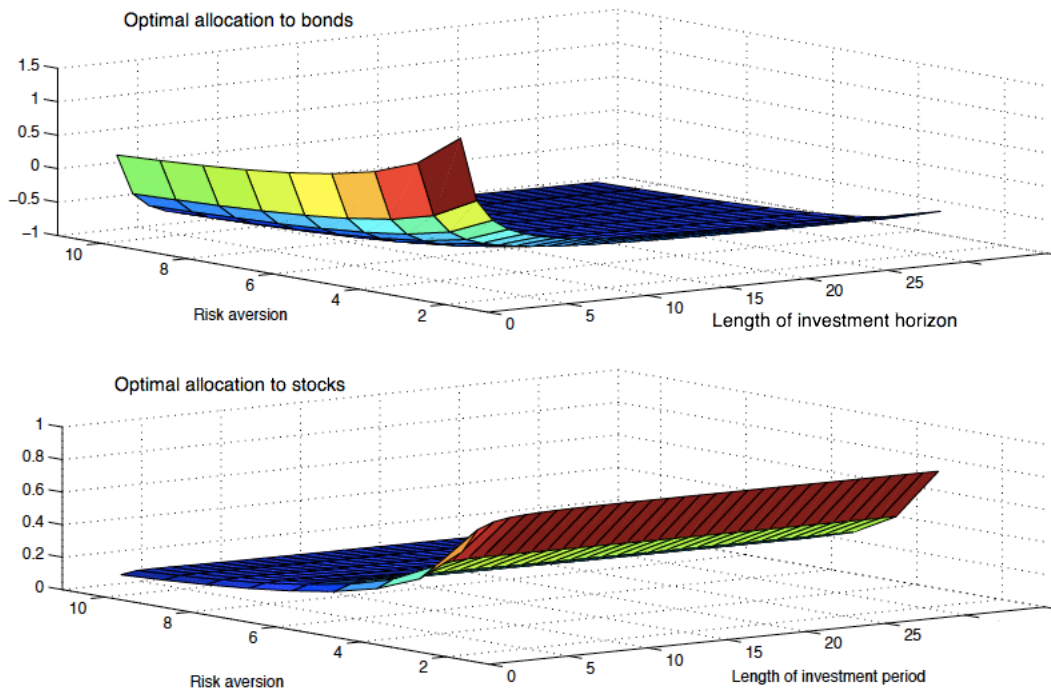


Figure 2.4: The figure illustrates the initial investments, that is, at $t = 0$, in bonds and stocks when the market is undervalued, and the length of the investment period increases. The length of the investment period ranges from zero years, $T = 0$, to twenty years, $T = 20$. The correlation between the predictor and the interest rate is assumed to be $\rho_{XB} = 0.3$ and the correlation between the stock price and the predictor is $\rho_{SX} = -0.8$.

Mean-Variance Portfolios

As an inspection of the trading policies given in Equation (2.26) shows, investments in the stock index is a linear function of the risk premium $\mu + \beta_S(X - \bar{X})$. Figure 2.5 illustrates the bond mean-variance portfolio under four different economic conditions. The mean-variance stock portfolio is constant in time and is represented as a dashed line in all panels of Figure 2.5. Panel *a*) illustrates the decomposed portfolios when the stock index is undervalued and the correlation between bonds and stocks is positive. The mean-variance stock portfolio is then positive, as the stock price is expected to decrease. The hedge demands for the predictor are both positive. Since the correlation is negative, the mean-variance bond portfolio is positive. Panel *b*) illustrates the case when the stock price is under valued and the bond-stock correlation is negative. The mean-variance stock portfolio is also positive in this case and because the bond-stock correlation is negative, the investor optimally keeps a short position in the bonds, as illustrated in the figure. Panels *c*) and *d*) illustrate the mean-variance portfolios when the stock price is overvalued. The mean-variance stock portfolio is in both cases negative, as the stock price is expected to decrease. The mean-variance bond portfolio depends on the bond-stock correlation, and is negative when the correlation is positive and vice versa. The mean-variance portfolios decrease monotonically in risk aversion.

Hedge Portfolios

The analysis of the mean-variance terms leads us to investigate how the hedge demands change over time and with different economic conditions. As explained earlier, as long as the correlation coefficients are nonzero, the bond portfolio hedges both interest rate risk and changes in the opportunity set, due to predictability. Figure 2.5 presents, among other things, how the hedge demand of the investor affects investments in both the bond and stock portfolio. When the spot level of the predictor is above the long-run mean, the investor takes advantage of this and increases investments in bonds through a positive hedge term for the predictor, as illustrated in panels *a*) and *b*) in Figure 2.5. This could also be denoted as a speculative term, in the sense that the investor invests more in bonds and more in stocks due to the fact that stock returns are predictable.

Both predictors move inversely to the stock price. This means that the higher the per share price, the lower the spot rate of the predictor. Thus, a low predictor spot rate may indicate an overvalued stock. This implies that if the spot level is below its long-run mean, the investor hedges against future changes in the stock-portfolio as a rise in the DPR might stem from a decrease in the stock price. The investor therefore increases the

optimal share in bonds at the expense of less exposure to stocks. The hedge demand for interest rate risk is negative throughout the investment period. As time passes, it becomes less and less important to hedge against shifts in the opportunity set and, eventually, at the investment horizon, the demand for hedging is zero. The stock portfolio hedges only shifts in the opportunity set due to the predictable variable, X . An illustration is given in Figure 2.5.

Because a spot rate of the predictor higher than the long-run mean may signal an undervalued stock index, the investor increases investments in the stock. The hedge demand for changes in opportunity set is therefore positive and can be denoted as speculative. The investor thus increases investments in the stock both through a higher mean-variance term, because the spot level, X , is present there, and through the speculative term. The speculative term decreases in time but, all else being constant, the mean variance term does not. This means that the investor decreases investments in stocks in time. The hedge demand decreases in time, as the investor sees no point in hedging against changes in the opportunity set after liquidating the portfolio. The more risk averse investor invests considerable less in the stock market, and the hedge demand decreases both absolute and relative to the optimal invested percentage in stocks.

Correlation between Bonds and Stocks

An important question is how the optimal investments are affected by the correlation between the asset classes. The bond-stock correlation is typically low, and is the reason why both bonds and stocks are included in a well diversified portfolio. There is, however, substantial time-variation in the bond-stock correlation. For instance, Buraschi et al. (2010) investigate how a stochastic correlation between asset classes affects optimal portfolio policies, and find that the hedge demand for correlation risk is substantial and about half the size of the volatility hedge demand. Several recent studies, for example Ilmanen (2003), Connolly et al. (2005), Barberis et al. (2005), and Christiansen and Rinaldo (2007), have found that the bond-stock correlation varies in time, and is negative in times of economic uncertainty, sometimes denoted as a *flight-to-quality*, where investors flock to assets with lower variability. In other market conditions, the correlation is found to be positive, see for example Shiller and Baltratti (1992) and Campbell and Ammer (1993). In this section, I will analyze how the optimal portfolios change with respect to the correlation coefficients between both asset classes and between the predictable variable and stock returns. The correlation between stocks and bonds is denoted ρ_{SB} and Equation (2.26) illustrates how this coefficient affects the trading policies.

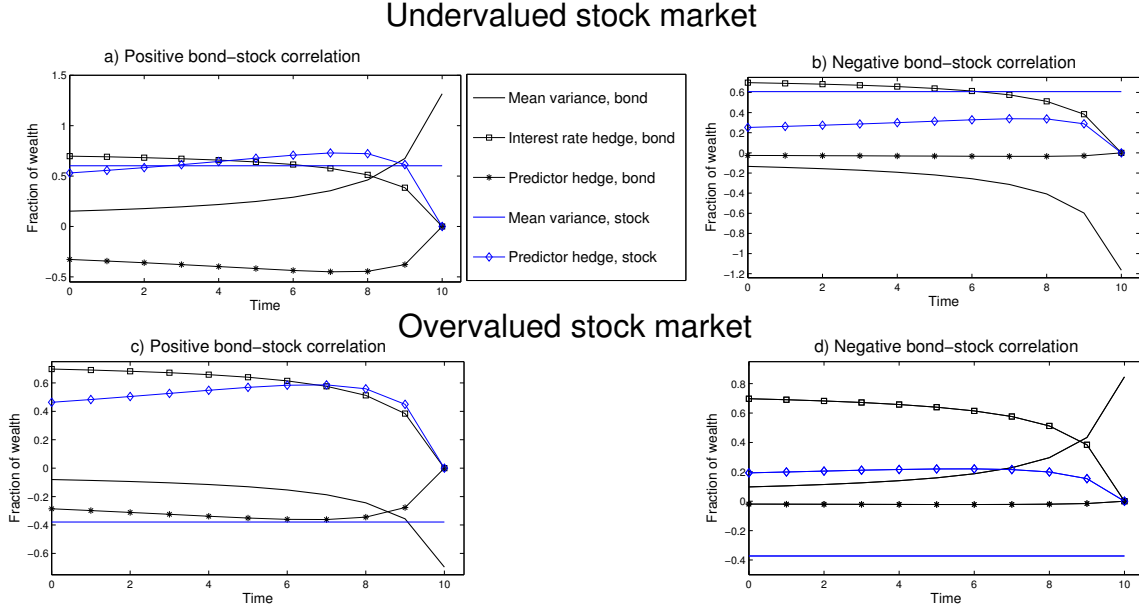


Figure 2.5: The figure illustrates the decomposition of the bond and stock portfolios under four different economic conditions. The hedge components converge to zero in time for both bonds and stocks. The bond mean-variance portfolio depends on time whereas the stock mean-variance is independent of time. The correlation between the predictor and the interest rate is assumed to be $\rho_{XB} = 0.3$ and the correlation between the stock price and the predictor is $\rho_{SX} = -0.8$. Risk aversion is assumed to be $\gamma = 3$.

Because stock returns are predictable through the LICR or the DPR, the stock price may be undervalued if the current level of the predictor is higher than the long-run mean. This means that the stock price is expected to increase over time as long as $X > \bar{X}$. In this case, I find that the investor is long in stocks, but the fraction invested in bonds depends strongly on the sign of the correlation coefficient. If the correlation is negative, then the investor is short in the bond, as the bond price is expected to decline. On the other hand, if the correlation is positive, then the investor is long in both bond and stock. I also find that when the correlation is positive, the investor reduces the fraction of wealth invested in the stock and hence reduces the risk of the portfolio through diversification.

Tables 2.2 and Table 2.4 illustrate the bond and stock portfolio when the stock market is undervalued, that is $\bar{X} < X$, for the LICR and the DPR, respectively. When the length of the investment period is short, then the interest rate hedge is negligible, but as the length of the investment period increases, the interest rate risk becomes larger and hence the interest rate hedge increases. This implies that when the correlation is positive, the investor will decrease investments in bonds as the length of the investment period

increases. At the same time, the investor increases investments in stocks. The percentage of wealth held in the bank account, or cash, increases as the investment period increases if the correlation is positive. If the correlation is negative, the fraction in the bank decreases as the investor decreases the short position in bonds, and increases the amount in stocks slightly.

Table 2.4: DPR and optimal asset allocation with varying bond-stock correlation when $\bar{X} < X$.

In this table, the level of the predictor is above its long-run mean, and specifically the predictor is equal to $X = \bar{X} + 2s$, where s is the sample standard deviation and equal 0.0167.

		ρ_{SB}				
T		- 0.3	-0.15	0	0.15	0.3
0	π^*	(-39.7, 28.3)	(-10.6, 26.0)	(15.4, 25.1)	(42.1, 25.4)	(73.5, 26.9)
	π_{mv}	(-39.7, 28.3)	(-10.6, 26.0)	(15.4, 25.1)	(42.1, 25.4)	(73.5, 26.9)
	π_{ih}	0	0	0	0	0
	π_{ph}	0	0	0	0	0
2	π^*	(-49.6, 43.6)	(-42.0, 42.1)	(-36.0, 43.6)	(-31.3, 48.7)	(-6.9, 42.1)
	π_{mv}	(-13.9, 28.3)	(-3.7, 26.0)	(5.4, 25.1)	(25.7, 25.4)	(25.7, 26.9)
	π_{ih}	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)
	π_{ph}	(-1.5, 15.2)	(-4.1, 16.1)	(-7.2, 18.5)	(-11.9, 23.3)	(1.5, 15.2)
5	π^*	(-29.8, 61.3)	(-33.0, 48.9)	(-32.7, 43.4)	(-30.7, 41.3)	(-27.3, 41.7)
	π_{mv}	(-7.5, 28.3)	(-2.0, 26.0)	(2.9, 25.1)	(7.9, 25.4)	(13.8, 26.9)
	π_{ih}	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)
	π_{ph}	(20.3, 32.9)	(11.6, 22.8)	(7.1, 18.3)	(4.0, 15.9)	(1.5, 14.9)
20	π^*	(-40.1, 47.0)	(-42.5, 39.6)	(-43.1, 36.3)	(-43.0, 35.2)	(-42.2, 35.9)
	π_{mv}	(-3.0, 28.3)	(-0.8, 26.0)	(1.2, 25.1)	(3.2, 25.4)	(5.5, 26.9)
	π_{ih}	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)
	π_{ph}	(11.5, 18.6)	(6.9, 13.6)	(4.3, 11.2)	(2.5, 9.8)	(0.9, 9.1)

2.2.8 Bond–Stock Ratio (BSR)

As analyzed in Section 2.2.7, and represented in Figures 2.2 and 2.3, the fraction invested in bonds and stocks may increase, or decrease in time. As a consequence, the bond–stock ratio (BSR), may also decrease or increase in time. The BSR is defined as

$$BSR(t) = \frac{\pi_B(t)}{\pi_S(t)},$$

Table 2.5: DPR and optimal asset allocation with varying bond-stock correlation when $\bar{X} > X$.

In this table, the level of the predictor is below its long-run mean, and specifically the predictor is equal to $X = \bar{X} - 2s$, where s is the sample standard deviation and equal 0.0167.

T		ρ_{SB}				
		- 0.3	-0.15	0	0.15	0.3
0	π^*	(26.6, -4.0)	(20.3 -4.1)	(15.4, -4.3)	(11.2, -4.7)	(7.2, -5.5)
	π_{mv}	(26.6, -4.0)	(20.3 -4.1)	(15.4, -4.3)	(11.2, -4.7)	(-7.2, -5.5)
	π_{ih}	0	0	0	0	0
	π_{ph}	0	0	0	0	0
2	π^*	(-5.8, 26.9)	(-16.7, 16.4)	(-22.6, 11.6)	(-26.8, 8.8)	(-30.4, 7.0)
	π_{mv}	(9.3, -4.0)	(7.1, -4.1)	(5.4, -4.3)	(3.9, -4.7)	(2.5, -5.5)
	π_{ih}	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)	(-34.2, 0)
	π_{ph}	(19.0, 30.8)	(10.4, 20.5)	(6.2, 15.9)	(3.4, 13.5)	(1.3, 12.4)
5	π^*	(-18.9 26.5)	(-28.3, 16.7)	(-33.4, 12.0)	(-37.0, 9.2)	(-40.0, 7.3)
	π_{mv}	(5.0, -4.0)	(3.8, -4.2)	(2.9, -4.3)	(2.1, -4.7)	(1.4 -5.5)
	π_{ih}	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)	(-42.7, 0)
	π_{ph}	(18.8, 30.5)	(10.6, 20.7)	(6.4, 16.3)	(3.6, 13.9)	(1.3, 12.8)
20	π^*	(-35.8, 13.6)	(-40.6, 8.6)	(-43.4, 6.0)	(-45.5, 4.2)	(-47.2, 2.7)
	π_{mv}	(2.0, -4.0)	(1.5, -4.2)	(1.2, -4.3)	(0.8, -4.7)	(0.5, -5.5)
	π_{ih}	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)	(-48.6, 0)
	π_{ph}	(10.8, 17.6)	(6.5, 12.7)	(4.0, 10.3)	(2.3, 9.0)	(0.8, 8.2)

The BSR represent how an investor should shift investments throughout the investment period. Typical advice given by investment professionals, such as Vanguard or Fidelity, is that young people should carry greater risk by leaning the portfolio towards stocks, and as the investment horizon approaches, the lions share of wealth should be invested in less risky bonds, thus the BSR should increase in time. These advisers also suggest that more risk averse investors should keep a larger BSR than less risk averse investors. However, as reported by Canner et al. (1997), finance theory suggest that the BSR should be independent of risk aversion, even though more risk averse investors should invest more in bonds than less risk averse investors. Canner et al. (1997) denoted the difference between theory and practice as the *asset allocation puzzle*. Several solutions has been proposed to this puzzle, see, for example Bajeux-Besnaiou et al. (2001), Brennan and Xia (2002), and intertemporal hedging has been proposed as a solution to this puzzle. However, as seen in panel *b* of Figure 2.3, the fraction invested in bonds may decrease in time whereas the fraction invested in stocks increase. This implies that the BSR will decrease in time as well.

This contradicts practical advice, but at the same time it makes sense for the investor, as panel *b* illustrates an undervalued stock market with a negative bond–stock correlation. Figure 2.6 gives an illustration of the BSR with respect to risk aversion, γ , and time, t . The investment horizon is assumed to be twenty years. As panels *b* and *d* of Figure 2.6 show, the BSR may be decreasing with respect to risk aversion, which contradicts standard financial theory. Panels *a*) and *c* of Figure 2.6 illustrates the BSR for an undervalued stock market. Because returns on stocks are expected to increase, investments in stocks increase whereas investments in bonds depend on the bond–stock correlation. However, the speed of mean reversion of the predictor implies that stock investments decrease towards the end of the investment horizon. The invested fraction in bonds depend on the bond–stock correlation. If negative, then the percentage in bonds is small but increasing in time. If positive, then the fraction is larger and at the same time, the fraction in stocks decrease compared to negative bond–stock correlation. This means that the investor smoothes the risk of the asset classes more evenly. In an overvalued stock market, the fraction in stocks is negative as returns are expected to be low, or negative. Depending on the bond–stock correlation, investments in bonds may be positive or negative, see panels *b* and *d* in Figure 2.6.

2.3 Conclusion

In this paper, I have investigated how predictability affects the portfolio choice problem of an investor with a power utility and a finite investment horizon. The investor chooses the optimal allocation between three asset classes: one risk-less and two risky. The risky asset classes are considered to be bonds and stocks. I find an analytical solution to the value function and the corresponding trading policies. By means of a verification theorem, I prove the optimality conditions of the value function. I find that the investment policies are time-varying and hedge demands in both bond and stock portfolios can be substantial. If the predictable variable is above its long-run mean and the correlation between bonds and stocks is positive, then the investor finances investments in stock by means of reducing investments in bonds. The predictor hedge demand can be negative or positive, depending on economic conditions.

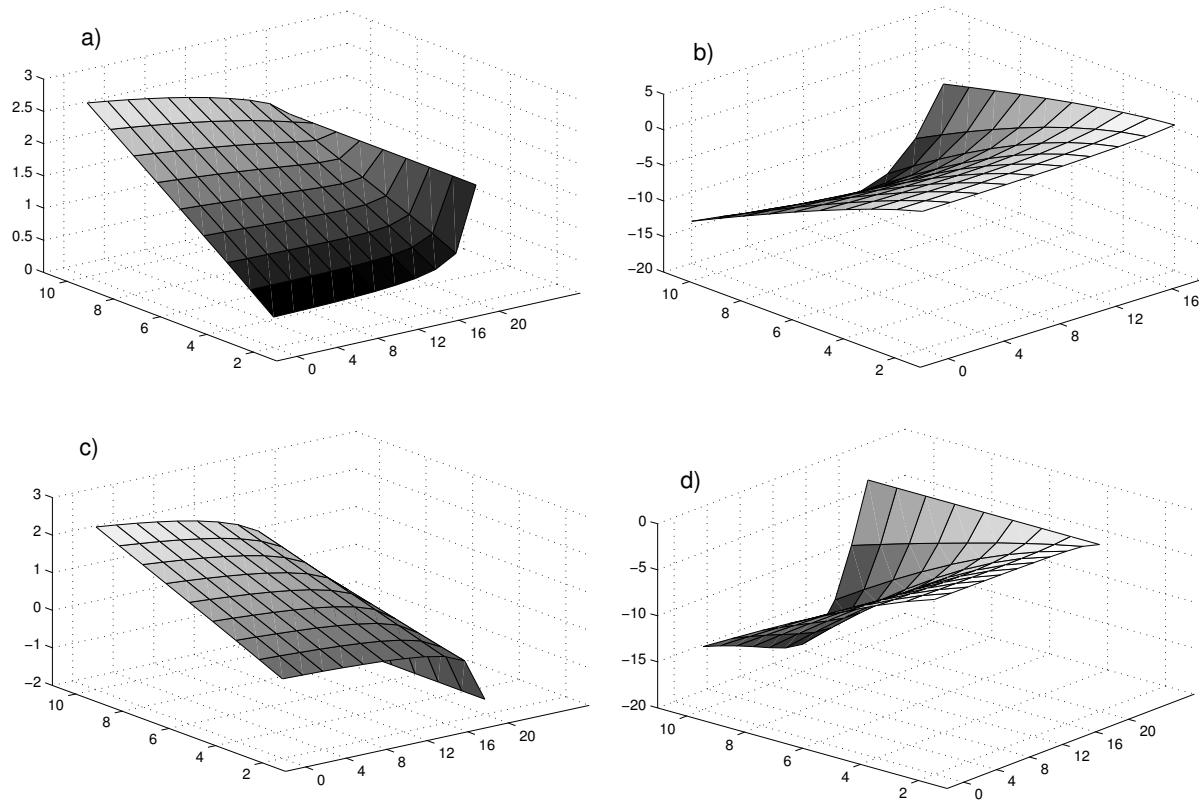


Figure 2.6: The BSR with respect to time and different economic conditions. Panels *a)* and *c)* corresponds to a stock market where the current DPR is above its long-run mean. Panels *b)* and *d)* corresponds current DPR below is long-run mean.

Chapter 3

Optimal Portfolio Choice under Time–Varying Interest Rates and Bid–Ask Spreads

Abstract

Standard models of liquidity suggest that the price of a risky asset increases in liquidity as it reflects future benefits the investors expect to receive. A time-varying liquidity therefore has an important impact on the price of the asset. Relying on transaction-level data for a broad panel of bonds from 2004 through 2012, I find that the estimated bid-ask spread of bonds are highly time-varying and mean-reverting. The spread estimator peaks in 2008 during the credit market crisis and has a substantial volatility in 2011 and 2012 during the sovereign debt crisis. I further show that the hedge demand for liquidity risk for finite horizon investors may be as much as 50 percent larger than the demand for hedging interest rate risk.

3.1 Introduction

The liquidity of bonds is widely known to have a significant impact on bond prices, see e.g. Edwards et al. (2007), Bao et al. (2011) and Amihud et al. (2013). Though there is a considerable number of papers analyzing liquidity and its effects on bond prices, there are few, if any, that analyze the dynamic optimal investment in bonds with time-varying liquidity risk. In this paper, I first compute a time-series of a particular measure of market liquidity, the bid-ask spread, and then quantify its impact on the optimal allocation of

wealth for a finite horizon investor. Applying transaction level data, I find that the bid-ask spread is mean-reverting with considerable volatility. This time-variation in liquidity has significant impact on the bond prices and hence on the optimal trading policies of a finite horizon investor.

When prevailing interest rates change, bond prices tend to move in response. This sensitivity to interest rates is the single greatest influence on bond prices. However, recent research has found that the liquidity of the bond also plays an important role in determining the price, see for example Amihud and Mendelson (1991), Bao et al. (2011) and Fontaine and Garcia (2012). These studies argue that liquidity is an important factor in determining bond prices. In particular, Bao et al. established a strong link between the liquidity and the price of bonds. The authors even found that changes in market-level liquidity explained a substantial part of time variations in the yield spreads of high-rated bonds, overshadowing the credit risk component. Although a precise definition of liquidity and its quantification will depend on the specific model considered, two properties are common. First, liquidity arises from market frictions, such as costs and constraints on trading and capital flows; second, its impact on the market is transitory. These two properties indicate that a stochastic process describing liquidity should capture a time variation around a long-run mean, see Chordia et al. (2000), Hasbrouck and Seppi (2001) and Acharya and Pedersen (2005). The high-low spread measure introduced by Corwin and Schultz (2012) is shown to possess these desired properties of market liquidity. As a matter of fact, Næs et al. (2011) gives a thorough review of several different liquidity measures, and documents that this mean-reverting feature is present in all of the measures considered. The literature on asset pricing often assumes an ideal world where security prices are set by market participants in frictionless financial markets. However, a variety of market frictions, including liquidity, trading costs¹ and constraints on short selling, exist in actual markets. It is important for market participants to accurately estimate and incorporate the impact of trading costs. For portfolio managers and investors, implementing investment decisions is costly and will typically lead to a shortfall in investment performance, see e.g. Perold (1988), relative to that theoretically attainable in frictionless markets. Decisions need to be conditioned not only on the fundamental soundness of potential investments, but also on the anticipated costs of implementing the required

¹Transaction costs are of two types: processing costs and market impact costs. Processing costs are commissions paid to brokers, taxes, and fees. These costs are known and easily measurable. Market impact costs consist of the spread and additional market impact, if the order exceeds the quote depth for the security to be purchased or sold. The market impact can be seen as the price concession we must pay to liquidity providers in order to accommodate our trade. Market impact represents the bulk of the total transaction costs, see e.g. Grinold and Kahn (1999).

trades.

In this paper, I compute the spread estimator for approximately 950 different corporate bonds traded in the period 2004 through 2012. The spread estimator is found to be highly time varying and mean reverting with a peak during the financial crisis of 2008 and significant increase in volatility in the sovereign debt crisis in 2011-2012. The size of the bid-ask spread of securities is one measure of the liquidity of the market, and if it is zero, it is said that the market is frictionless. The trader initiating the transaction is said to demand liquidity and the other party is said to supply liquidity, thus the difference in price bid by the buyer and the price asked by the seller is the liquidity cost. The literature distinguishes between two types of liquidity; funding and market liquidity, see Brunnermeier and Pedersen (2009). This paper focus on the the bid-ask spread, which measures the ease of trading an asset and hence is a measure of market liquidity. I show that the bid-ask spread is uncorrelated with the interest rate.

I assume that the bond liquidity, as measured by the bid-ask spread, can be represented as the continuous time mean reverting process described by Vasicek (1977). I then apply the method described in Duffie and Singleton (1999) to price the bond when the interest rates and liquidity determine the price of the bond. With the dynamical expression of the bond price, I describe the dynamical evolution of the wealth of the investor. From this expression I formulate the Hamilton-Jacobi-Bellmann Equation from which I propose a solution to the optimal investment problem. The investor is assumed to have a constant relative risk aversion, or CRRA preferences, and her investment horizon is finite. The investor will allocate wealth between a riskless asset, also denoted money market account, and the bond.

A central question is, therefore: How should an investor allocate assets when time-varying market liquidity affects bond prices? This is important since increases in the liquidity, which means a decrease in the bid-ask spread, lead to increases in the price of the bond. On the other hand, if the liquidity decreases, then the bond price also decreases. This will have several implications for how the investor optimally invests in the risky asset since bond prices will vary not only with respect to the interest rate, but also the level of the liquidity.

Since I focus on the bid-ask spread, the estimator is non-negative, and the larger the value of the estimator, the lower the liquidity of the bond. While a large body of literature analyzes the pricing of assets when the liquidity is time-varying, little research explores optimal dynamic trading strategies and optimal portfolio choice under liquidity risk. The aim of this paper is to correct this deficiency and to analyze how liquidity affects

the allocation of a portfolio and the corresponding trading strategies. The investor has some positive initial wealth, w_0 , which she wants to allocate so that she can optimize the expected utility of the wealth at the investment horizon. There is no consumption from the portfolio during the investment period, so all gains and losses stem from the trading policies. Under these assumptions, I then derive analytical expressions for the optimal portfolio policies. I find that the trading policy can be decomposed into three distinct components. One component is usually denoted the mean-variance portfolio, as it reflects the investments if there is no intertemporal hedging. The other two components describe the demand for hedging interest rate risk and liquidity risk, respectively. Moreover, I find that the mean variance component and the hedging term for liquidity risk both depend on the spot level of the liquidity. The implication is that not only does the investor hedge for an expected level of liquidity, but she is also taking into account the current level of liquidity, hence there is an element of market timing in the optimal trading policies. If the current level of the liquidity indicates a higher level than the long-run mean, then the price of the bond is lower than the expected returns. Thus, I find that due to the increased yield on the bond, the investor optimally increases the amount invested. On the other hand, higher liquidity implies lower expected returns since the price of the bond is higher than expected. The investor will, in this case, optimally reduce the share of wealth, and sometimes even keep a short position in the bond.

In order to find a closed form solution to the indirect value function, I need to reduce the dimension of the corresponding Hamilton-Jacobi-Bellman equation. I achieve this through a Cole-Hopf transformation similar to Zariphopoulou (1999) and Benth et al. (2001). Through this transformation most of the non-linear terms in the HJB equation vanish, and I am left with an equation for which a solution exists. The result is an analytical solution to the optimal portfolio choice problem and the corresponding optimal trading policy. Unlike previous research, this is the first attempt to solve analytically an optimal portfolio choice problem for bonds when liquidity is an endogenous stochastic process.

3.2 Data Description and Summary

The main dataset applied for this paper is TRACE, Transaction Reporting and Compliance Engine, maintained by the Financial Industry Regulatory Authority, FINRA. This dataset is a result of recent regulatory initiatives to increase the price transparency in secondary corporate bond markets. Data from TRACE has been widely used to analyze

bond prices, see e.g. Edwards et al. (2007) and Bao et al. (2011). FINRA is responsible for operating the reporting and dissemination facility for over-the-counter corporate bond trades. On July 1, 2002, FINRA began Phase I of bond transaction reporting, requiring that transaction information be disseminated for investment grade securities with an initial issue size of \$1 billion or greater. Phase II, implemented on April 14, 2003, expanded reporting requirements, bringing the number of bonds to approximately 4,650. In this paper I will use a subset of the total amount of bonds reporting prices to TRACE. The bonds I consider are issued by the companies listed on the Dow Jones Industrial Average index as of December 31st 2012. In total I have prices on 947 bonds from January 2nd 2004 through December 31 2012. Of course, new issuances and retired bonds generate some time variation in the cross-section of bonds in the sample. Some bonds are traded intensely when issued and shortly after, but then *goes away*, a terminology used for bonds that are kept by the investor.

In order for a bond to be included in the data sample, there must be observed values of high and low prices during the day of trading. This means that if there is no trade in a particular bond, then it will not affect the estimated bid-ask spread. Of course, the reason why the bond is not traded may be that the asked price is so far from the offer price, that the investors does not come to an agreement. However, the estimator applied in this paper utilizes daily high and low prices, and thus it is necessary that the bonds are traded in order to be included in the sample. This is a drawback of the bond data, as the TRACE database only contain reported prices of trades, an hence if there are no trades, there are no observations. There are $N = 947$ bonds in the sample, all of which are investment grade with an average credit rating of AA from Standard & Poor's. A AA rating implies that the obligor has a very strong capacity to meet its financial commitments. It differs from the highest rated obligors, AAA, only in small degree. That the bonds are on average rated as high grade bonds is not surprising since the bonds are all issued by companies listed in the Dow Jones Industrial Average Index, an index that shows how 30 large publicly owned companies based in the United States have traded during a standard trading session in the stock market. In addition I will apply data on the interest rate over the same time-period as for the bid-ask spread. The level of the interest rate is illustrated in Figure 3.1.

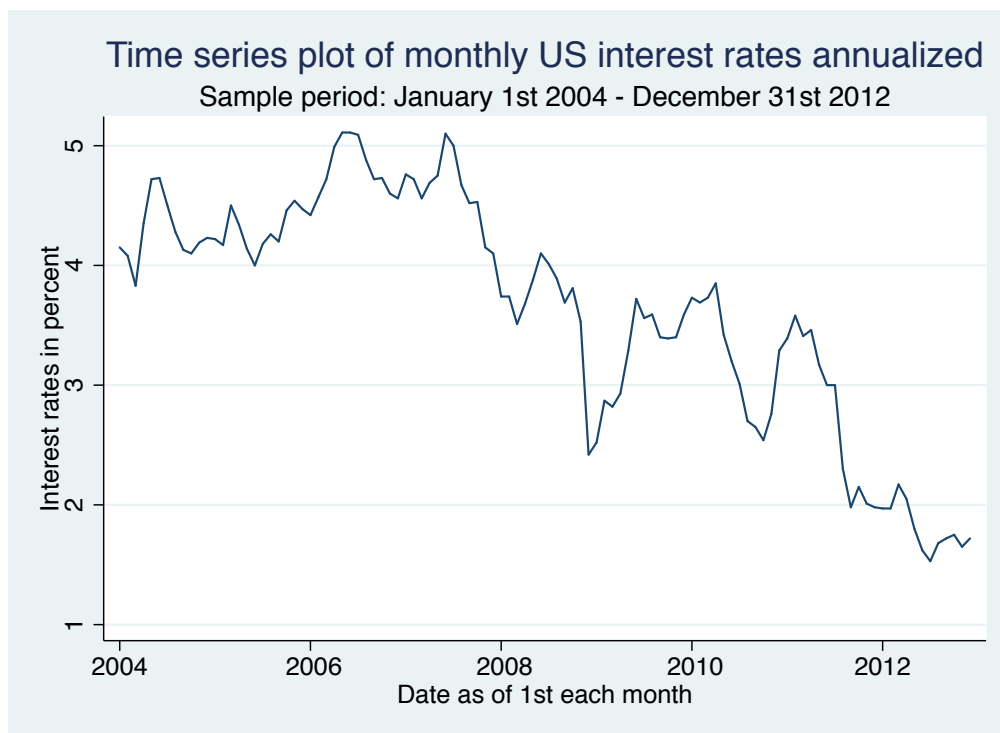


Figure 3.1: The Figure shows a time series of the interest rate in the United States from January 1st 2004 through December 31st 2012. The data is obtained from Robert Shiller: www.yale.econ.edu/shiller. The sample mean is found to be 3.63 whereas the sample standard deviation of the time series is found to be 0.99.

Table I represents the estimates of the parameters of the equations applied in this paper.

Table I
Parameter estimates

This table presents the estimates of the parameters in the models describing the interest rate and the liquidity. In order to estimate the process describing the bid-ask spread of bonds, I have applied observations on bond trades obtained from the Transaction Reporting and Compliance Engine Information, TRACE. Observations about the interest rate and the stock market is obtained from Robert Shiller, <http://www.econ.yale.edu/~shiller/>. Correlation between the interest rate and the liquidity is found to be negligible.

Process	Parameter description	Notation	Parameter values
Interest rate	Speed of mean reversion	κ_r	0.0279
"	Long-run mean	\bar{r}	0.0392
"	Volatility	σ_r	0.0179
Liquidity	Speed of mean reversion	κ_l	0.3675
"	Long-run mean	\bar{l}	0.6035
"	Volatility	σ_l	0.2711
Correlation	interest rate - liquidity	ρ	-0.06

3.3 The Bid-Ask Spread

The bid-ask spread is widely accepted to capture the current market liquidity of any risky asset, see e.g. Amihud and Mendelson (1986), Conroy et al. (1990), Huang and Stoll (1997) and Corwin and Schultz (2012). In the bond market, bond prices are quoted in pairs: the *bid* and the *ask*. The difference between the bid and the ask is known as the *spread*. This spread is the difference between what a dealer pays to buy a bond and the price at which he wants to sell it. Thus technically, the bid is what you sell for and the ask is what you buy for. The size of the spread reflects what is known as a bond's *liquidity*; that is, the ease and cost of trading a particular bond. A narrow spread indicates high demand and low risk whereas a wide spread indicates an unwillingness on the part of a dealer to own a bond without a substantial price cushion. Spreads and liquidity vary based on the bond market in which bonds trade. Treasuries are by far the most liquid of all bonds, and consequently sell at the narrowest spreads. Spreads and liquidity also vary over time. In strong markets spreads tend to narrow and in weak markets they widen. During the credit crisis of 2008, spreads widened so far beyond the norm that many bonds could not be sold at any price. Amihud and Mendelson (1986) establishes the theory on the effect of liquidity on asset

values and provides estimations of the relation between expected returns and liquidity across different assets. The authors found that portfolio returns increase with the level of the bid-ask spread, showing that investors ask in equilibrium to be compensated for higher trading costs. Spreads vary widely², see Pedersen (2009), Corwin and Schultz (2012) and Foucault et al. (2013), both in the cross-section and in time series, and one of the chief factors determining the size of the spread is the demand for a particular bond, that is, how easy it is to sell. If you are selling an inactively traded bond, then the broker makes sure that she buys it cheaply enough so that she will not lose money when she resells. Spreads typically ranges from 0.25% to 1% for actively traded Treasuries and to as much as 4% for inactively traded bonds. The relationship between the bid-ask spread and returns is found to be increasing and concave, as the study by Amihud and Mendelson (1986) shows. The authors found that for stocks that are traded on the American Stock Exchange, a bid-ask spread of 0.5% implies an excess monthly return of approximately 0.3%, a bid ask spread of 1% approximately 0.45% excess returns and a bid-ask spread of 3% implies an excess return of approximately 0.7%. The spread varies for many reasons, e.g. the type of bond, the number of bonds being sold, the bonds maturity, it's credit quality, the interest rates, demand for a specific bond and so forth.

3.3.1 The Bid-Ask Spread Estimator

In order to measure the liquidity of bonds, I rely on the bid-ask spread estimator derived in the paper by Corwin and Schultz (2012). This estimator relies is a function of daily high and low prices over one-day and two-day intervals. The high-low spread estimator is derived under general conditions and it is simple to compute. Corwin and Schultz finds that the high-low spread estimator outperforms alternative low-frequency spread estimators. Denote H_t^o as the highest observed price at day t and L_t^o as the lowest observed price at day t . Let β_t denote the sum of the squared gross difference between the high and low observed price over two consecutive days:

$$\beta_t = \sum_{i=0}^1 \left[\ln \left(\frac{H_{t-i}^o}{L_{t-i}^o} \right) \right]^2 \quad (3.1)$$

Since $H_t > L_t$, β_t will be a sum of two squared numbers that are both greater than zero. Let ξ_t denote the squared gross difference between the two-day maximum and minimum

²See for example Acharya and Pedersen (2005) for a study of another liquidity measure and the impact of its time-variability on the pricing of assets. Also, Bessembinder and Venkataraman (2009) measures the execution costs in financial markets applying the bid-ask spread.

of the observed prices:

$$\xi_t = \left[\ln \left(\frac{H_{t-1,t}^o}{L_{t-1,t}^o} \right) \right]^2 \quad (3.2)$$

Since ξ_t is computed over a two-day interval, the nominator and denominator might be from two different days, e.g. $\xi_t = \left(\ln \left(\frac{H_t}{L_{t-1}} \right) \right)^2$. Let α denote the proportional difference between the square root of ξ and β defined by:

$$\alpha_t = (1 + \sqrt{2})(\sqrt{\beta_t} - \sqrt{\xi_t}) \quad (3.3)$$

The spread estimator is then given by

$$S_t = \frac{2(e_t^\alpha - 1)}{e_t^\alpha + 1} \quad (3.4)$$

For small values of α , the estimate of the spread, S , will be approximately the same as α itself. Since α is computed from observed values, it will have substantial time-variation. On drawback of this bid-ask estimator is that ξ_t might be larger than β_t , hence it sometimes produces a negative bid-ask spread, clearly a violation of reality. In the stock market this is often due to after-hours trading, as reported in Corwin and Schultz (2012). To compute the spread estimator for bonds, I apply observations from the TRACE database described above. Figure 3.2 show a time series of the mean estimate of the bid-ask spread for bonds issued by the companies currently listed on the Dow Jones Industrial Average index.

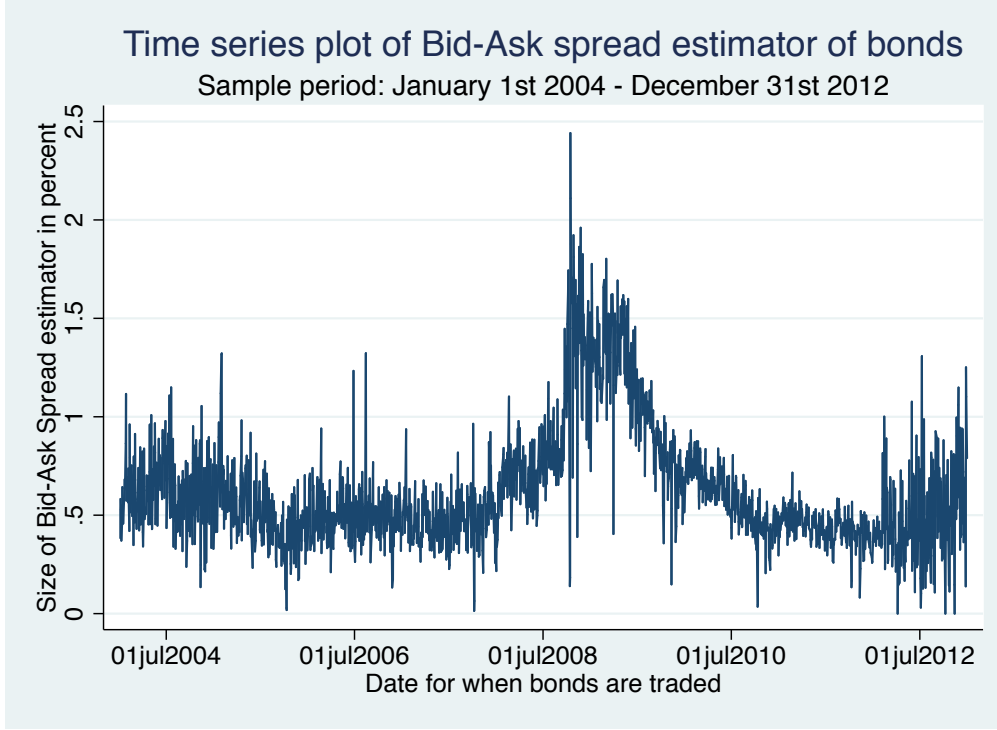


Figure 3.2: The Figure shows a time series of the bid-ask spread estimator given in Equation (3.4). The estimator is varying around a mean of 0.006, which translates into a 0.6% spread. The estimator is clearly mean-reverting with considerable variation. The spike in the spread in 2008 corresponds to the credit crisis at that time, whereas the increased volatility in 2011 and 2012 corresponds to the period of the sovereign debt crisis. There are 2205 observations of the bid-ask spread, with a sample average $\bar{S} = 0.60$ and standard deviation equal to 0.32.

The bid-ask spread is clearly varying in time with a mean for which it evolves around. The long-run mean of the bid-ask spread is fairly constant, but it has a spike around 2008 during the financial crisis. The highest estimated value of the bid-ask spread, 2.44%, is found October 15th 2008, occurs the day after the US announces a plan to take an equity interest of \$250 billion in the largest banks in the US. October 15th the Dow Jones Industrial Average suffered its largest drop in terms of percentage since 1987, falling over 733 points. I assume that the bid-ask spread, S , of bonds can be represented by the Vasicek (1977) model

$$dl_t = \kappa_l(\bar{l} - l_t)dt - \sigma_l dZ_l \quad (3.5)$$

where κ_l , \bar{l} and σ_l are positive constants and $Z_l = (Z_l(t))_{t \geq 0}$ is a standard Brownian motion. I estimate the parameters of the process given in Equation (3.5) using the time

series of the bid-ask spread illustrated in Figure 3.2. I find that the long-run mean is $\bar{l} = 0.6025$, the volatility is $\sigma_l = 0.2711$ and the speed of mean reversion equals $\kappa_l = 0.3675$. These estimates are used to quantify the impact of the liquidity on the optimal allocation in bonds of a finite horizon investor.

As Figure 3.3 shows, the correlation between the interest rate and the bid-ask spread is negligible. The bid-ask spread represented in the scatterplot is averaged over each month in the sample, so there are 108 observations.

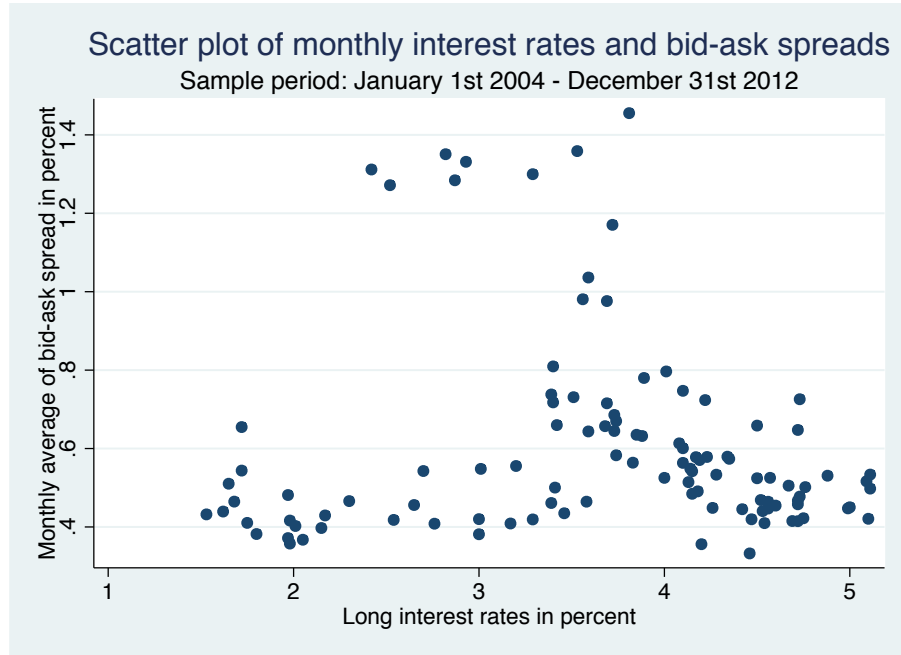


Figure 3.3: The Figure shows a scatter plot of the monthly long interest rates and the monthly average bid-ask spread. The correlation between the two processes is found to be -0.06.

3.4 Description of the Model

I model the investment choice of a price-taking individual who can trade in bonds and an instantaneous risk-free asset. Throughout the paper, I consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \leq T}, P)$ where $(\mathcal{F})_{t \leq T}$ is the filtration at time t , that is it contains all information up to time t . I assume that the risk-free asset, also denoted as the bank account, serves as the numeraire so that all asset prices are specified in units of this asset.

3.4.1 Financial Assets

I assume that the short-term interest rate, r_t , follows the Vasicek (1977) model,

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r dZ_r \quad (3.6)$$

where κ_r , \bar{r} , and σ_r , are positive constants and $Z_r = (Z_r(t))_{t \geq 0}$ is a standard Brownian motion under the probability measure P and independent of Z_l . The interest rate is recognized as the main driver of bond prices. However, several recent studies has pointed to the role of liquidity and how the impact of time-varying market liquidity affects the prices of bonds, see for example Duffie and Singleton (1999), Chen et al. (2007), Fontaine and Garcia (2012), Bao et al. (2011), and Amihud et al. (2013). Following Duffie and Kan (1996), Duffie and Singleton (1999), and Dai and Singleton (2000), the price of a zero-coupon bond at time t with maturity at time T_1 is then given by

$$B(t, T_1, r, l) = \mathbb{E}_Q^t \left[\exp \left(- \int_t^{T_1} R_s ds \right) \right] \quad (3.7)$$

where $R_s = r_s + l_s$ and $\mathbb{E}_Q^t[X] = \mathbb{E}_Q[X|\mathcal{F}_t]$ describes the expectation of X under the risk neutral probability Q conditional on the information at time t . Duffie and Singleton (1999) consider $R_s = r_s + l_s + h_s$, where h_s is interpreted as the default rate. In this paper, I focus on investment grade and government bonds and hence I assume that the probability of default is zero. This is a reasonable assumption for a significant number of countries in Asia, western Europe, Canada,³ and Australia, which all have a *AAA* rating, yet the bonds issued by countries with a *AAA* rating, have very different liquidity. For example, US government bonds are the most liquid bonds available, yet US keeps only a *AA+*, one level below *AAA*. Further, the paper by Amihud and Mendelson (1991) shows that there is a significant liquidity premium for even the most liquid assets in the world. The estimated bid-ask spread for the bonds discussed in Section 2.1 will serve as a benchmark for the market liquidity of bonds applied in the rest of the paper. For this reason, I remove the default rate probability, h_s , yet keep the liquidity measure, l_s , to price the bonds. So, after taking expectations, the price of the bond is given by⁴

$$B(t, T_1, r_t, l_t) = \exp (A_1(t, T_1) - A_2(t, T_1)r_t - A_3(t, T_1)l_t). \quad (3.8)$$

³Countries in Asia with *AAA* rating includes Hong Kong and Singapore. In Europe; Denmark, Finland, Germany, Liechtenstein, Luxembourg, Netherlands, Norway, Sweden, Switzerland and United Kingdom all have a *AAA* rating, according to Standard & Poor's, a rating agency.

⁴A formal derivation is given in the Appendix, see Proposition B.1.1.

The function $A_1(t, T_1)$ is given by

$$\begin{aligned}
A_1(t, T_1) = & \left[\left(\left(\frac{\sigma_r^2}{2\kappa_r^2} - \bar{r} \right) + \left(\frac{\sigma_l^2}{2\kappa_l^2} - \bar{l} \right) \right) (T_1 - t) + \left(\bar{r} - \frac{\sigma_r^2}{\kappa_r^2} \right) A_2(t, T_1) \right. \\
& + \left(\bar{l} - \frac{\sigma_l^2}{\kappa_l^2} \right) A_3(t, T_1) + \frac{\sigma_r^2}{4\kappa_r^3} (1 - e^{-2\kappa_r(T_1-t)}) \\
& \left. + \frac{\sigma_l^2}{4\kappa_l^3} (1 - e^{-2\kappa_l(T_1-t)}) + \int_t^{T_1} \sigma_B(t, T_1) \lambda ds \right] \quad (3.9)
\end{aligned}$$

where $\sigma_B(t, T_1) > 0$ is the bond price volatility and given by $[-\sigma_r A_2(t, T_1), -\sigma_l A_3(t, T_1)]$. The functions $A_2(t, T_1)$ and $A_3(t, T_1)$ are given by

$$A_2(t, T_1) = \frac{1}{\kappa_r} (1 - e^{-\kappa_r(T_1-t)}) \quad (3.10)$$

$$A_3(t, T_1) = \frac{1}{\kappa_l} (1 - e^{-\kappa_l(T_1-t)}). \quad (3.11)$$

The functions $A_i(t, T_1)$ are deterministic and converge to zero as $t \rightarrow T_1$, which implies that the price of the bond equals one at maturity. Following Langetieg (1980), the dynamics of the price $B(t, T_1)$ of such a bond is given by⁵

$$\frac{dB(t, T_1, r_t, l_t)}{B(t, T_1)} = (r_t + l_t + \sigma_B(t, T_1) \lambda) dt + \sigma_B(t, T_1) dZ. \quad (3.12)$$

The market price of risk is given by $\lambda = [\lambda_r, \lambda_l]^\top$, is assumed deterministic and $dZ = [dZ_r, dZ_l]^\top$. This implies that the volatility of the bond depends on the time to maturity, but not the level of the interest rate or the level of the bid-ask spread.

3.4.2 Investor Preferences

I assume throughout the paper that the investor has a time-additive utility function of terminal wealth w_T . Thus, given the opportunity to invest in riskless and risky assets, the investor starts with a positive initial wealth, w_0 , and chooses, at each time t , $0 \leq t \leq T$, to invest a fraction, π_t , of the wealth in the risky asset so as to maximize the expected wealth

$$\max_{\pi_t, 0 \leq t \leq T} \mathbb{E} \left[U(w_T) \right]. \quad (3.13)$$

⁵A formal derivation is given in the Appendix, see Proposition B.1.2 and Proposition B.1.3.

Here, T is the investment horizon,⁶ assumed nonrandom and where the wealth satisfies the self-financing condition given by the following result:

$$dw_t = \left(r + \pi_t(l + \sigma_B(t, T_1)\lambda) \right) w_t dt + \pi_t w_t \sigma_B(t, T_1) dZ. \quad (3.14)$$

Although the model could be extended to allow for intermediate consumption, I use this simpler specification to focus more directly on the intuition behind the results. Throughout the paper, I use a power utility function

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}$$

where $\gamma > 0$ is the constant relative risk aversion.

3.4.3 Optimal Portfolio Choice

I will now turn to the portfolio choice problem of Equations (3.13) and (3.14). I will apply the dynamic programming principle when solving for the optimal portfolio strategy, and hence use the stochastic control approach. Following standard literature on the matter, for example Merton (1971), I define the indirect utility function as

$$V(t, w, r, l) = \max_{\pi_t, 0 \leq t \leq T} \mathbb{E} \left[U(w_T) \right]. \quad (3.15)$$

The expectation is computed given the values of w , r , and l at time t and given the strategy π_t . This leads to the corresponding HJB equation for the conjectured value function V , which in this case is given by

$$\begin{aligned} 0 = & \max_{\pi} \left\{ V_t + (r + \pi_t(l + M_1))wV_w + \kappa_r(\bar{r} - r)V_r + \kappa_l(\bar{l} - l)V_l \right. \\ & \left. + \frac{w^2}{2}\pi_t^2 M_2 V_{ww} + \frac{\sigma_r^2}{2}V_{rr} + \frac{\sigma_l^2}{2}V_{ll} - \pi_t w M_3 V_{rw} - \pi_t w M_4 V_{lw} \right\}, \end{aligned} \quad (3.16)$$

where the subscripts on V denote a partial derivative with respect to that particular variable. The coefficients M_i are given by:

$$M_1(t) = -\sigma_r A_2(t, T_1)\lambda_r - \sigma_l A_3(t, T_1)\lambda_l \quad (3.17)$$

$$M_2(t) = \sigma_r^2 A_2(t, T_1)^2 + \sigma_l^2 A_3(t, T_1)^2 \quad (3.18)$$

⁶A constant subjective discount term is omitted as there is no consumption from the portfolio, and the interest rate r_t discounts future wealth, see Ingersoll (1987), Korn and Kraft (2001), and Puhle (2007).

$$M_3(t) = \sigma_r^2 A_2(t, T_1) \quad (3.19)$$

$$M_4(t) = \sigma_l^2 A_3(t, T_1) \quad (3.20)$$

This means that $M_i(t)$ depend on time, t , and the maturity of the bond, T_1 . The functions $M_i(t)$ will be scaled by the volatility of either the interest rate, σ_r , or the liquidity, σ_l . To find an analytical solution to the HJB equation, I will make the simplifying assumption:

Assumption 3.4.1 *The following ratios are constant*

$$\frac{M_3^2(t)}{M_2(t)} = K_1, \quad \frac{M_4^2(t)}{M_2(t)} = K_2$$

The function $M_2(t)$ is positive for all values of t and hence the constants K_1 and K_2 are both positive. Because $M_3(t)$ and $M_4(t)$ are squared, the volatilities σ_r and σ_l will be taken to the power of four, and as these coefficients are small in absolute terms, they will dominate the ratios and K_1 and K_2 will be small. Note that the investment horizon is $T < T_1$, hence both nominator and denominator will always be positive and there is no risk of dividing on zero.

In the following result, I solve for the optimal portfolio strategy, π_t^* , by first conjecturing that the indirect utility function is given by a specific form, and then verifying that it is the optimal candidate for the value function.

Proposition 3.4.1 *Let K_1 and K_2 be constants. Then the value function can be separated into the product*

$$V(t, w, r, l) = \frac{w_t^{1-\gamma}}{1-\gamma} f^{\theta_1}(t, r) h^{\theta_2}(t, l) \quad (3.21)$$

with the terminal condition

$$V(T, w, r, l) = \frac{w_T^{1-\gamma}}{1-\gamma}.$$

The functions $f(t, r)$ and $h(t, l)$ are given by

$$\begin{aligned} f(t, r) &= \exp(g_1(t) + g_2(t)r) \\ h(t, l) &= \exp(g_3(t)l + g_4(t)l^2) \end{aligned}$$

and the optimal trading policy is given by

$$\begin{aligned}
\pi_t^* &= \frac{1}{\gamma M_2} \left[l_t + M_1 - \theta_1 M_3 g_2(t) - \theta_2 M_4 (g_3(t) + 2g_4(t)l_t) \right] \\
&= \frac{l_t + M_1}{\gamma M_2} - \frac{\theta_1 M_3 g_2(t)}{\gamma M_2} - \frac{\theta_2 M_4 (g_3(t) + 2g_4(t)l_t)}{\gamma M_2} \\
&= \pi_{mv}^* + \pi_r^* + \pi_l^*
\end{aligned} \tag{3.22}$$

Proof is found in the Appendix.

To reduce the dimension of the HJB equation, I make use of a technique similar to the one applied Zariphopoulou (1999) and Benth and Karlsen (2005). I find that if the functions f and h both have exponents, equal to θ_1 and θ_2 , and when these two coefficients have a particular form, some of the nonlinear terms in the HJB equation (3.16) cancel.⁷ I find that if the coefficients are equal to

$$\begin{aligned}
\theta_1 &= \frac{\sigma_r^2}{\sigma_r^2 + \frac{1-\gamma}{\gamma} K_1} \\
\theta_2 &= \frac{\sigma_l^2}{\sigma_l^2 + \frac{1-\gamma}{\gamma} K_2}.
\end{aligned}$$

then there is an analytical solution to Equation (3.16). Further, the ratio $\frac{1-\gamma}{\gamma}$ is negative for all values of the risk aversion and will be a number between negative one and zero. For reasonable choices of parameters, the products $\frac{1-\gamma}{\gamma} K_1$ and $\frac{1-\gamma}{\gamma} K_2$ will be less than σ_r^2 and σ_l^2 , respectively. This implies that the power coefficients θ_1 and θ_2 will always be positive. If there is no stochastic liquidity, that is, $\sigma_l = 0$, then the coefficient K_1 reduces to σ_r^2 , which implies that the coefficient θ_1 simplifies to γ . This coincides with the findings of Zariphopoulou (1999), where the author reduces the dimension of the HJB equation in a portfolio consisting of bonds with stochastic interest rates. Proposition 3.4.1 shows that for investors with a risk aversion γ , the optimal investments in the risky asset changes in time. Specifically, Equation (3.22) shows that the optimal portfolio share in the risky bond includes the time-varying liquidity spot rate, l . In addition, the investment horizon plays a significant role through the functions $g_i(t)$. The functions, $g_i(t)$, are converging to zero in time and all are equal to zero at the investment horizon, $t = T$. The specific expressions for these functions are found in the proof of the Proposition. The maturity of the bond will also affect the trading strategy through the coefficients, M_i , which equal zero at the time T_1 . However, because the investment horizon comes before the maturity

⁷In the papers by Zariphopoulou (1999) and Benth and Karlsen (2005) only one function f was included in the indirect utility function, and the exponent was equal to the constant risk aversion, γ .

of the bond, there is no risk of dividing by zero in the optimal portfolio policies.

3.4.4 Portfolio Decomposition

The portfolio can be decomposed into three distinct terms: 1) the mean-variance term, π_{mv}^* , 2) the hedge demand for interest rate risk, π_{ih}^* , and last, 3) the hedge demand for illiquidity risk, π_{lh}^* . Equation (3.22) represents the decomposition of the optimal portfolio. All portfolios depend on time, t , and the risk aversion, γ . At the investment horizon, the investor has no motivation to hedge future changes in the opportunity set, as the portfolio is liquidated and the investor will start consuming the accumulated wealth, which implies that both hedge components, π_{ih}^* and π_{lh}^* , will equal zero. Figure 3.4 illustrates the optimal share of wealth invested in the bond, π^* , with respect to time and risk aversion.

Optimal fraction of wealth invested in bonds

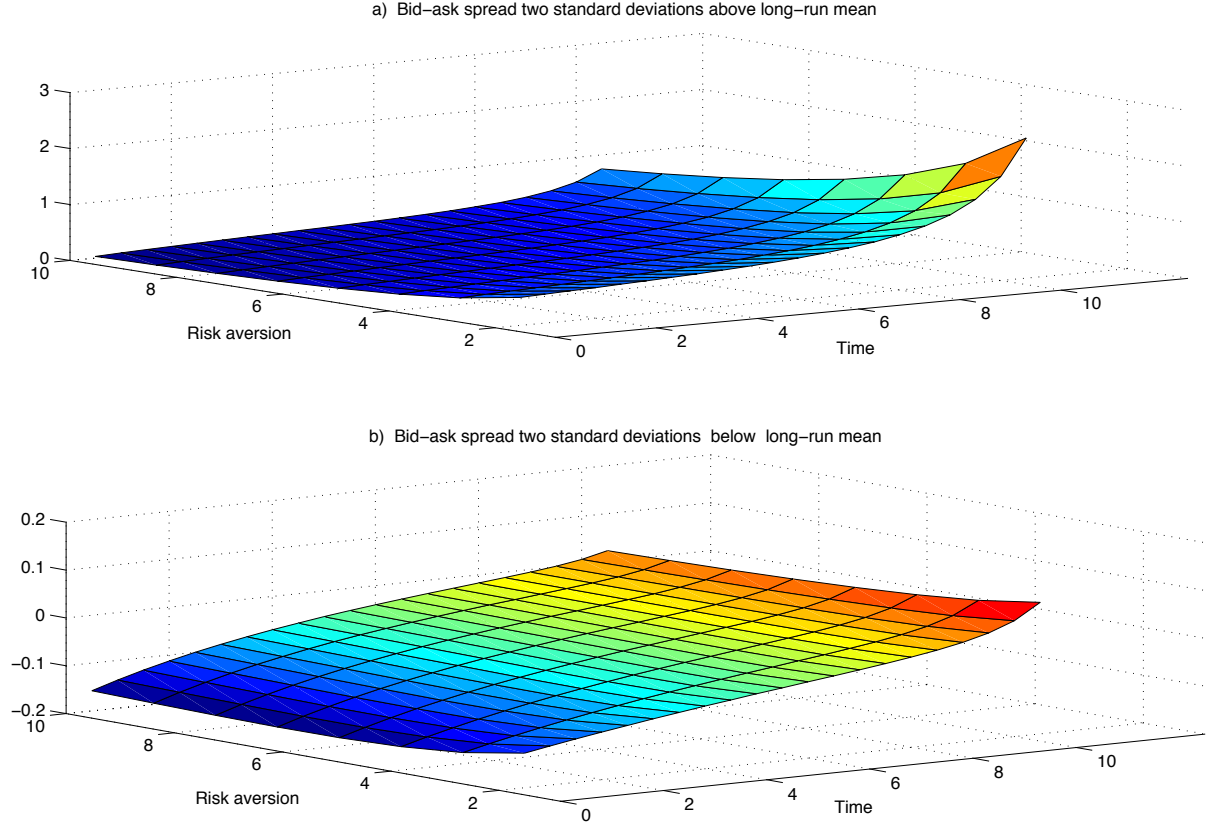


Figure 3.4: This figure illustrates the optimal share invested in the bond with respect to time and risk aversion. The investment period is ten years and the spot level of the liquidity is $l_t = \bar{l} \pm 2$ standard deviations. Panel a) represents a bid-ask spread of 1.15 percent whereas panel b) represents a bid-ask spread of 0.06 percent. The long-run mean of the liquidity is $\bar{l} = 0.60$. The volatility of the liquidity is $\sigma_l = 0.27$.

As Figure 3.4 illustrates, the optimal share of wealth invested in the bond decreases in risk aversion and increases in time no matter the level of the bid-ask spread. However, if the bid-ask spread is above the long-run mean, which implies higher expected returns as the liquidity is low and the corresponding bond price is low, the optimal fraction of wealth invested in bonds is always positive. If, on the other hand, the bid-ask spread is narrow, and liquidity is high, then the corresponding expected returns are low due to the higher

bond price, and hence the investor might optimally short sell the bond.

Mean-Variance Portfolio

The mean-variance portfolio varies in time; in particular, it increases in time as the investor buys more bonds in the end of the investment period. The initial investment in bonds depends on the length of the investment period, but the investment at the horizon does not. This means that for $t = T$, $\pi_{mv}(T)$ is the same no matter whether $T = 1, 2$, or 20 . Figure 3.5 illustrates the mean-variance component of the optimal portfolio. The mean-variance term includes the spot level of the illiquidity, l_t , directly, so there is an element of market timing in this portfolio. Panel *a*) in Figure 3.5 represents the mean variance portfolio when the current level of the bid-ask spread is above the long-run mean. This implies that the bond is less liquid than usual and hence it is reflected in a low price and hence higher expected returns. The investor therefore optimally increases the fraction in bonds. As time to the horizon approaches, the mean-variance portfolio increases for all levels of risk aversion. This is true for all risk preferences of the investor, as the relative change from the beginning of the investment period to the end is the same for all investors. This fact is seen if one divides the mean-variance portfolio at the beginning of the investment period by the mean-variance portfolio at the end of the investment period: $\frac{\pi_{mv}(0)}{\pi_{mv}(T)}$. This ratio is independent of γ , so the relative change will always be the same for investors of all risk preferences. Panel *b*) of Figure 3.5 represents the mean variance component when the bid-ask spread is narrower than the long-run mean, that is, liquidity is high. The mean-variance portfolio is then negative and hence the investor reduces investments in the bond as expected returns are low.

Hedge Portfolios

The investor hedges change in the opportunity set due to both interest rate risk and liquidity risk. Because changes in the opportunity set have a potential impact on the expected return of the portfolio before the investment horizon, and not after, the hedge demands equals zero at the investment horizon, that is, when $t = T$. The presence of liquidity magnifies the optimal investment in the bond. The sign of the correction depends on the level of the bid-ask spread and whether it is above or below the long-run mean. I find that the liquidity risk hedge demand may exceed the demand for hedging interest rate risk. As Figure 3.6 shows, the size of the liquidity hedge component can be as much as 50 percent higher, in absolute terms, than the hedge component for interest rate risk. The main reason for this is the relatively high volatility of the bid-ask spread

estimator compared to the volatility of the interest rate. Both hedge components are relatively unaffected by the risk aversion. The liquidity hedge portfolio reflects how the size of the spot level of the bid-ask spread affects the demand for hedging future changes in the opportunity set due to changes in the bid-ask spread. Both hedge portfolios increase in absolute terms in the length of the investment horizon.

3.4.5 Allocation with Respect to Liquidity

Assets with high liquidity trade with a premium because they offer the option to sell easily at later stages in the investment period. Assets with low liquidity do not offer the possibility to easily sell, thus these assets trade with a discount. This implies that the expected rate of return is higher when liquidity is low. Because the investor does not consume from the portfolio and has a finite investment horizon, the optimal portfolio will consist of a short position in liquid assets and a long position in illiquid assets. Figure 3.7 illustrates these properties. In the calculations in this section, I consider a risk aversion of three and an investment horizon of ten years, whereas the maturity of the bond is eleven years. The rest of the parameters are as described in table 3.2. If the current bid-ask spread is wide, that is, $l_t > \bar{l}$, then the investor optimally increases the share in this bond. If the liquidity remains low, the optimal fraction in the bond increases towards the investment horizon. On the other hand, if the bond is liquid ($l_t < \bar{l}$) expected returns are lower, and the investor optimally reduces the exposure to this bond, but increases the fraction in bonds as the investment horizon approaches. For a very high market liquidity, indicated by a narrow bid-ask spread ($l_t < 0.5$ percent), the investor is optimally short in the bonds. As the investment horizon approaches, the investor optimally invest a positive amount in the bonds. In the frictionless case, that is, a bid-ask spread of 0 percent, the investor will keep a short position in the bond throughout the investment period. This is because the bank account pays the same interest, but without any risk.

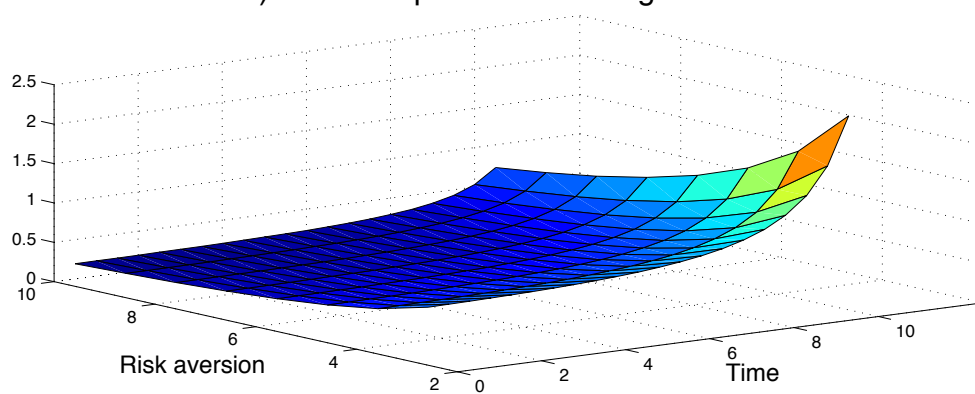
3.4.6 Conclusion

I analyze the impact of time-varying liquidity, as measured by the bid-ask spread, on the optimal allocation of a portfolio of bonds. I find the indirect utility function and the corresponding optimal trading policy in closed form. First, I decompose the trading strategy into three components: the mean-variance component, an interest rate hedging component, and a liquidity risk hedging component. I find that all these three components are time-dependent. The two hedging terms converge to zero in time, and equal zero at the investment horizon. Moreover, I find that the spot level of the bid-ask spread is included

in the optimal trading strategy. The investor will therefore not only be concerned about the expected level of liquidity, as is the case for the interest rate, but also the current level and whether it is higher or lower than the long-run mean of the bid-ask spread. I find that when the bid-ask spread is lower than the long-run mean, it can be optimal to keep a short position in the bond. This is because a narrow bid-ask spread implies high liquidity and hence low expected returns. In this case, the investor will decrease the share of wealth invested in the bond with time. In fact, the marginal decrease is decreasing as well. The hedge demands for interest rate risk and liquidity risk are different in size. I find that the liquidity hedge demand can be as much as 50 percent larger than the interest rate hedge demand in the beginning of the investment period. An interesting extension would be to include inflation as well, because an important question is whether it is better to roll over highly liquid, and thus expensive, short-term bonds, or instead, to trade cheaper bonds with a longer horizon, but which carry the risk of inflation to a higher degree.

Mean variance portfolios

a) Bid-ask spread above long-run mean



b) Bid-ask spread below long-run mean

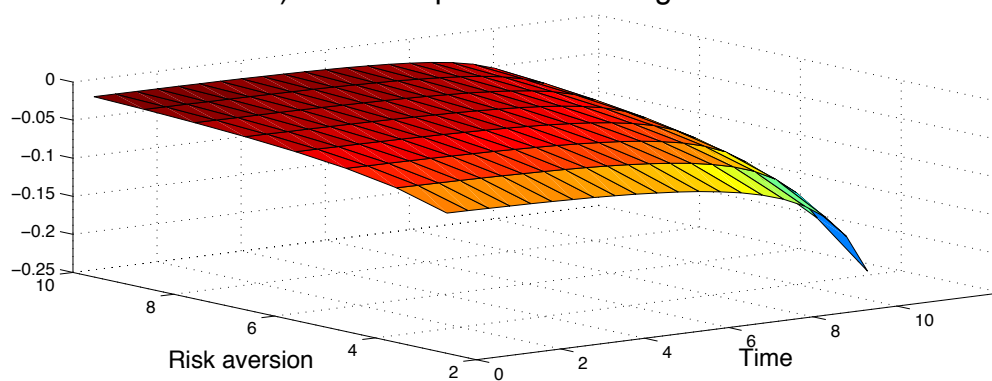


Figure 3.5: This figure illustrates the optimal mean variance portfolio with respect to time and risk aversion. The investment period is 10 years and the spot level of the liquidity is $l_t = 1.15$ percent, and the long-run mean of the liquidity is $\bar{l} = 0.60$. The volatility of the liquidity is $\sigma_l = 0.27$, whereas the volatility of the interest rate is $\sigma_r = 0.018$. The market prices of risk are assumed to be $\lambda_r = 0.03$ and $\lambda_l = 0.03$ for interest rate and illiquidity, respectively.

The hedge portfolios

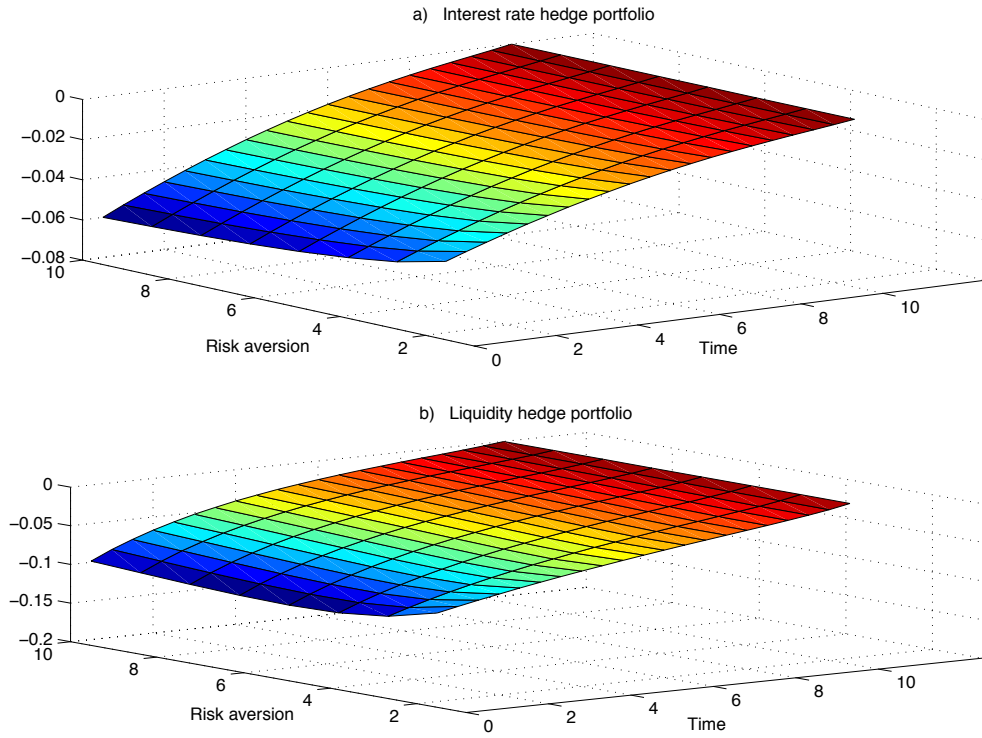


Figure 3.6: This figure illustrates the optimal hedge portfolios with respect to time and risk aversion. The investment period is 10 years and the spot level of the liquidity is $l_t = 1.15$ percent and the long-run mean of the volatility is $\bar{l} = 0.60$. The volatility of the liquidity is $\sigma_l = 0.27$, whereas the volatility of the interest rate is $\sigma_r = 0.018$. The market prices of risk are assumed to be $\lambda_r = 0.03$ and $\lambda_l = 0.03$ for interest rate and illiquidity, respectively.

Optimal allocation with respect to time and liquidity

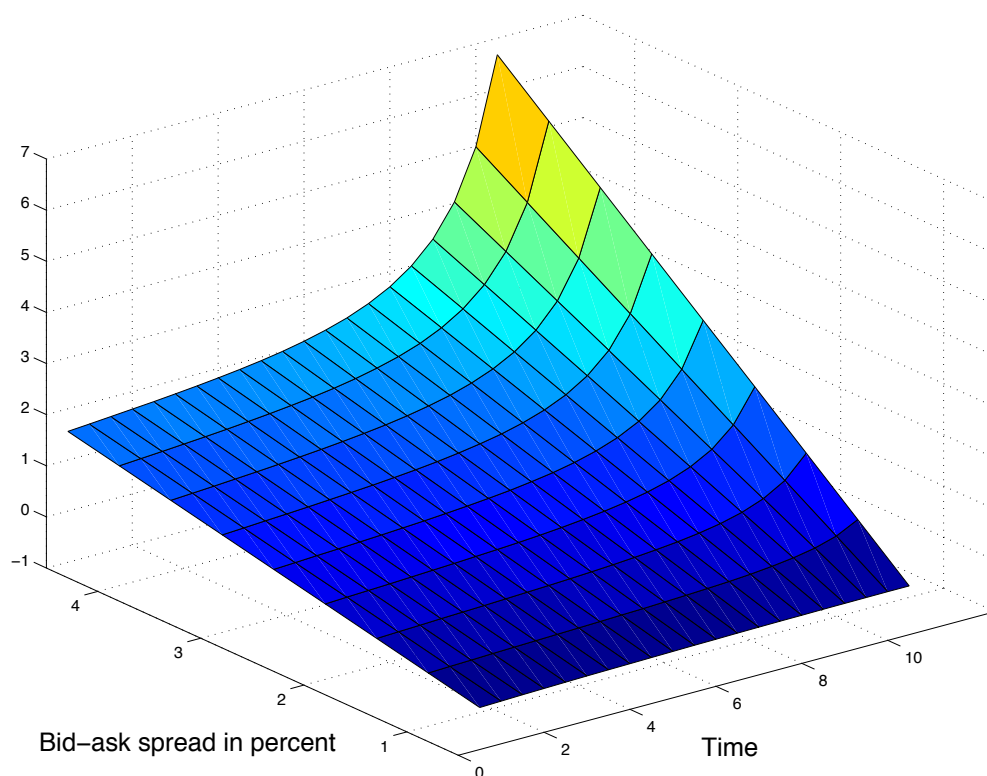


Figure 3.7: This figure illustrates the optimal fraction of wealth invested in the bond with respect to the liquidity of the bond over the entire investment period. The share increases in the width of the bid-ask spread at all times. For a given level of illiquidity, the optimal share increases in time, but more rapidly when the bid-ask spread is far above the long-run mean, indicating a low liquidity and hence high expected returns. The risk aversion is set to $\gamma = 3$, the investment horizon is ten years and the maturity of the bond is eleven years.

Appendices

Appendix A

Proofs for and Complimentary Results for Chapter 1

A.1 Proof of Proposition 2.2.3:

Let

$$V(t, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma} f(t, X)^{\theta_1} g(t, r)^{\theta_2}. \quad (\text{A.1})$$

then

$$\begin{aligned} V_t &= \left[\theta_1 \frac{f_t}{f} + \theta_2 \frac{g_t}{g} \right] V \\ V_w &= \frac{1-\gamma}{w} V, \quad V_{ww} = (-\gamma) \frac{1-\gamma}{w^2} V \\ V_X &= \frac{\theta_1 f_X}{f} V, \quad V_{XX} = \theta_1 \left[(\theta_1 - 1) \frac{f_X^2}{f^2} + \frac{f_{XX}}{f} \right] V \end{aligned}$$

$$\begin{aligned} V_r &= \frac{\theta_2 g_r}{g} V, \quad V_{rr} = \theta_2 \left[(\theta_2 - 1) \frac{g_r^2}{g^2} + \frac{g_{rr}}{g} \right] V \\ V_{rX} &= \theta_1 \theta_2 \frac{f_X g_r}{f g} V, \quad V_{wX} = \frac{\theta_1 (1-\gamma)}{w} \frac{f_X}{f} V \\ V_{wr} &= \frac{\theta_2 (1-\gamma)}{w} \frac{g_r}{g} V \end{aligned}$$

which implies that

$$\begin{aligned}
\pi_t &= -\frac{\frac{1-\gamma}{w}V}{w(-\gamma)\frac{1-\gamma}{w^2}V} \left[(\Sigma_t^\top)^{-1}\Lambda \right] + \frac{\frac{\theta_2(1-\gamma)}{w}\frac{f_X}{f}V}{w(-\gamma)\frac{1-\gamma}{w^2}V} \sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 \\
&\quad - \frac{\frac{\theta_1(1-\gamma)}{w}\frac{f_X}{f}V}{w(-\gamma)\frac{1-\gamma}{w^2}V} \sigma_X(\Sigma^\top)^{-1}\rho_{XP} \\
&= \frac{(\Sigma_t^\top)^{-1}\Lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 + \frac{\theta_1 f_X}{\gamma f} \sigma_X(\Sigma^\top)^{-1}\rho_{XP}
\end{aligned}$$

Substituting these expressions into the HJB Equation (2.19), I get the following, term by term:

1)

$$V_t = \left[\theta_1 \frac{f_t}{f} + \theta_2 \frac{g_t}{g} \right] V$$

2)

$$\begin{aligned}
(r_t + \Sigma_t \pi_t \lambda) w V_w &= (r_t + \Sigma_t \pi_t \lambda) w \frac{1-\gamma}{w} V \\
&= (r_t + \Sigma_t \pi_t \lambda) (1-\gamma) V \\
&= r(1-\gamma) V + \Sigma_t \left[\frac{(\Sigma_t^\top)^{-1}\lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 \right. \\
&\quad \left. + \frac{\theta_1 f_X}{\gamma f} \sigma_X(\Sigma^\top)^{-1}\rho_{XP} \right] (1-\gamma) V
\end{aligned}$$

3)

$$\begin{aligned}
\frac{1}{2} w^2 \pi_t^\top \Sigma_t \Sigma_t^\top \pi_t V_{ww} &= \frac{1}{2} w^2 \pi_t^\top \Sigma_t \Sigma_t^\top \pi_t (-\gamma) \frac{1-\gamma}{w^2} V \\
&= \frac{-\gamma}{2} \pi_t^\top \Sigma_t \Sigma_t^\top \pi_t (1-\gamma) V \\
&= \frac{-\gamma}{2} \left[\frac{(\Sigma_t^\top)^{-1}\lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 + \frac{\theta_1 f_X}{\gamma f} \sigma_X(\Sigma^\top)^{-1}\rho_{XP} \right] \\
&\quad \cdot \Sigma_t \Sigma_t^\top \left[\frac{(\Sigma_t^\top)^{-1}\lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 + \frac{\theta_1 f_X}{\gamma f} \sigma_X(\Sigma^\top)^{-1}\rho_{XP} \right] \\
&= \frac{-\gamma}{2} \left[\frac{\lambda^\top \lambda}{\gamma^2} - 2 \frac{\theta_2 g_r \sigma \lambda^\top \mathbf{e}_1}{\gamma^2 g} + 2 \frac{\theta_1 f_X \sigma_X \lambda^\top \rho_{XP}}{\gamma^2 f} + \frac{\theta_2^2 g_r^2 \sigma^2 \mathbf{e}_1^\top \mathbf{e}_1}{\gamma^2 g^2} \right. \\
&\quad \left. - 2 \frac{\theta_2 g_r \sigma \theta_1 f_X \sigma_X}{\gamma^2 g f} \mathbf{e}_1^\top \rho_{XP} + \frac{\theta_1^2 f_X^2 \sigma_X^2}{\gamma^2 f^2} \rho_{XP}^\top \rho_{XP} \right] (1-\gamma) V \\
&= \left[-\frac{\lambda^\top \lambda}{2\gamma} + \frac{\theta_2 g_r \sigma \lambda^\top \mathbf{e}_1}{\gamma g} - \frac{\theta_1 f_X \sigma_X \lambda^\top \rho_{XP}}{\gamma f} - \frac{\theta_2^2 g_r^2 \sigma^2 \mathbf{e}_1^\top \mathbf{e}_1}{2\gamma g^2} \right. \\
&\quad \left. + \frac{\theta_2 g_r \sigma \theta_1 f_X \sigma_X}{\gamma g f} \mathbf{e}_1^\top \rho_{XP} - \frac{\theta_1^2 f_X^2 \sigma_X^2}{2\gamma f^2} \rho_{XP}^\top \rho_{XP} \right] (1-\gamma) V
\end{aligned}$$

4)

$$\kappa(\bar{X} - X)V_X = \kappa(\bar{X} - X)\frac{\theta_1 f_X}{f}V$$

5)

$$\kappa_r(\bar{r} - r)V_r = \kappa_r(\bar{r} - r)\frac{\theta_2 g_r}{g}V$$

6)

$$\frac{\sigma_r^2}{2}V_{rr} = \frac{\sigma_r^2}{2}\theta_2\left[(\theta_2 - 1)\frac{g_r^2}{g^2} + \frac{g_{rr}}{g}\right]V$$

7)

$$\frac{\beta_X^2}{2}V_{XX} = \frac{\sigma_X^2}{2}\theta_1\left[(\theta_1 - 1)\frac{f_X^2}{f^2} + \frac{f_{XX}}{f}\right]V$$

8)

$$-\sigma_r\sigma_X\rho_{XB}V_{rX} = -\sigma_r\sigma_X\rho_{XB}\theta_1\theta_2\frac{f_X g_r}{fg}V$$

9)

$$\begin{aligned} \sigma_X\pi_t^\top\Sigma_t\rho_{XP}wV_{wX} &= \sigma_X\pi_t^\top\Sigma_t\rho_{XP}w\frac{\theta_1(1-\gamma)}{w}\frac{f_X}{f}V \\ &= \sigma_X\left[\frac{(\Sigma_t^\top)^{-1}\lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g}\sigma_r(\Sigma^\top)^{-1}\mathbf{e}_1 + \frac{\theta_1 f_X}{\gamma f}\sigma_X(\Sigma^\top)^{-1}\rho_{XP}\right]^\top \\ &\quad \cdot \Sigma_t\rho_{XP}\theta_1(1-\gamma)\frac{f_X}{f}V \\ &= \left[\sigma_X\rho_{XP}\theta_1\frac{\lambda^\top}{\gamma}\frac{f_X}{f} - \frac{\theta_2 g_r}{\gamma g}\sigma_r\mathbf{e}_1^\top\sigma_X\rho_{XP}\theta_1\frac{f_X}{f} \right. \\ &\quad \left. + \frac{\theta_1^2 f_X^2}{\gamma f^2}\sigma_X^2\rho_{XP}^\top\rho_{XP}\right](1-\gamma)V \end{aligned}$$

10)

$$\begin{aligned}
-\sigma_r \pi_t^\top \Sigma \mathbf{e}_1 w V_{wr} &= -\sigma_r \pi_t^\top \Sigma \mathbf{e}_1 w \frac{\theta_2(1-\gamma)}{w} \frac{g_r}{g} V \\
&= -\sigma_r \left[\frac{(\Sigma_t^\top)^{-1} \lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r (\Sigma^\top)^{-1} \mathbf{e}_1 \right. \\
&\quad \left. + \frac{\theta_1 f_X}{\gamma f} \sigma_X (\Sigma^\top)^{-1} \rho_{XP} \right]^\top \Sigma_t \mathbf{e}_1 \theta_2 (1-\gamma) \frac{g_r}{g} V \\
&= \left[-\sigma_r \frac{\lambda^\top}{\gamma} \mathbf{e}_1 \theta_2 (1-\gamma) \frac{g_r}{g} + \frac{\theta_2^2 \sigma_r^2 g_r^2}{\gamma g^2} \mathbf{e}_1^\top \mathbf{e}_1 \right. \\
&\quad \left. - \frac{\theta_1 \theta_2 f_X g_r}{\gamma f g} \sigma_X \rho_{XP}^\top \mathbf{e}_1 \right] (1-\gamma) V
\end{aligned}$$

Some of these terms contain highly nonlinear terms, such as $\frac{f_X^2}{f^2}$ and $\frac{g_r^2}{g^2}$. To find a solution to the HJB equation, it is necessary that these terms vanish, and it is here that the expressions of θ_i plays its role. For example, terms with $\frac{f_X^2}{f^2}$ are found in expressions 3, 7 and 9. If these terms are collected and factorized, the following is received

$$\begin{aligned}
0 &= \frac{\theta_1^2 f_X^2}{\gamma f^2} \beta_X^2 \rho_{XP}^\top \rho_{XP} (1-\gamma) + \frac{\beta_X^2}{2} \theta_1 (\theta_1 - 1) \frac{f_X^2}{f^2} - \frac{\theta_1^2 f_X^2 \beta_X^2}{2 \gamma f^2} \rho_{XP}^\top \rho_{XP} (1-\gamma) \\
&= \beta_X^2 \frac{f_X^2}{2 \gamma f^2} \left[2 \theta_1^2 \rho_{XP}^\top \rho_{XP} (1-\gamma) + \theta_1 (\theta_1 - 1) \gamma - \theta_1^2 \rho_{XP}^\top \rho_{XP} (1-\gamma) \right] \\
&= \beta_X^2 \frac{f_X^2}{2 \gamma f^2} \left[\theta_1^2 \rho_{XP}^\top \rho_{XP} (1-\gamma) + \theta_1^2 \gamma - \theta_1 \gamma \right] \\
&= \beta_X^2 \frac{f_X^2}{2 \gamma f^2} \left[\theta_1^2 \left(\rho_{XP}^\top \rho_{XP} (1-\gamma) + \gamma \right) - \theta_1 \gamma \right]
\end{aligned}$$

This implies that if θ_1 is given by

$$\theta_1 = \frac{\gamma}{\rho_{XP}^\top \rho_{XP} (1-\gamma) + \gamma} \quad (\text{A.2})$$

then all expressions that contain $\frac{f_X^2}{f^2}$ cancel. Further, from terms 3, 6, and 10, the expressions with the nonlinear expression $\frac{g_r^2}{g^2}$ vanish if $\theta_2 = \gamma$. Further,

$$\Sigma_t = \begin{bmatrix} \sigma_B(t) & 0 \\ \sigma_S \rho_{SB} & \sigma_S \sqrt{1 - \rho_{SB}^2} \end{bmatrix}.$$

The expression of Σ_t implies that

$$\Sigma_t^{-1} = \frac{1}{\sigma_B(t)\sigma_S\sqrt{1-\rho_{SB}^2}} \begin{bmatrix} \sigma_S\sqrt{1-\rho_{SB}^2} & 0 \\ -\sigma_S\rho_{SB} & \sigma_B(t) \end{bmatrix}.$$

so

$$(\Sigma_t^{-1})^\top = \frac{1}{\sigma_B(t)\sigma_S\sqrt{1-\rho_{SB}^2}} \begin{bmatrix} \sigma_S\sqrt{1-\rho_{SB}^2} & -\sigma_S\rho_{SB} \\ 0 & \sigma_B(t) \end{bmatrix} = (\Sigma^\top)^{-1}.$$

This implies that $(\Sigma_t^{-1})^\top \Lambda$ is

$$\begin{aligned} (\Sigma_t^{-1})^\top \Lambda &= \frac{1}{\sigma_B(t)\sigma_S\sqrt{1-\rho_{SB}^2}} \begin{bmatrix} \sigma_S\sqrt{1-\rho_{SB}^2} & -\sigma_S\rho_{SB} \\ 0 & \sigma_B(t) \end{bmatrix} \begin{bmatrix} \lambda_r \\ \lambda_S + \hat{\mu} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_r}{\sigma_B(t)} - \frac{\rho_{SB}\lambda_S}{\sigma_B(t)\sqrt{1-\rho_{SB}^2}} - \frac{\rho_{SB}\hat{\mu}}{\sigma_B(t)\sqrt{1-\rho_{SB}^2}} \\ \frac{(\lambda_S + \hat{\mu})}{\sigma_S\sqrt{1-\rho_{SB}^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_r\sigma_S - (\mu + \beta_S(X - \bar{X}))\rho_{SB}}{\sigma_B(t)\sigma_S(1-\rho_{SB}^2)} \\ \frac{\mu + \beta_S(X - \bar{X}) - \rho_{SB}\sigma_S\lambda_r}{\sigma_S^2(1-\rho_{SB}^2)} \end{bmatrix} \end{aligned}$$

Hence, the portfolio, π_t , is given by

$$\begin{aligned} \pi_t &= \frac{(\Sigma_t^\top)^{-1}\Lambda}{\gamma} - \frac{\theta_2 g_r}{\gamma g} \sigma_r (\Sigma^\top)^{-1} \mathbf{e}_1 + \frac{\theta_1 f_X}{\gamma f} \beta_X (\Sigma^\top)^{-1} \rho_{XP} \\ &= \begin{bmatrix} \frac{\lambda_r\sigma_S - \mu\rho_{SB}}{\sigma_B(t)} \\ \frac{\mu - \rho_{SB}\sigma_S\lambda_r + \beta_S(X - \bar{X})}{\sigma_S(1-\rho_{SB}^2)} \end{bmatrix} - \frac{\sigma_r g_r}{\sigma_B(t)g} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_X \frac{\theta_1 f_X}{\gamma f} \begin{bmatrix} \frac{\rho_{XB} - \rho_{SB}\rho_{XS}}{\sigma_B(t)(1-\rho_{SB}^2)} \\ \frac{\rho_{XS} - \rho_{SB}\rho_{XB}}{\sigma_S(1-\rho_{SB}^2)} \end{bmatrix} \end{aligned}$$

Let $g(t, r) = \exp(h_0(t) + h_1(t)r)$ and $f(t, X) = \exp(h_2(t)X + h_3(t)X^2)$ and substitute this into Equation (2.19). Multiply all terms with $1 - \gamma$, and the HJB equation is given by, after collecting all terms with equal expression of the state variable:

$$\begin{aligned}
0 = & \theta_2 \left[h'_0 + \frac{\kappa_X \bar{X} \theta_1}{\theta_2} h_2 + \left(\frac{\lambda_r^2}{2\gamma} + \frac{\lambda_S^2}{2\gamma} + \lambda_S \frac{\beta_S \bar{X}}{\gamma \sigma_S \sqrt{1 - \rho_{SB}^2}} + \frac{\beta_S^2 \bar{X}^2}{2\gamma \sigma_S^2 (1 - \rho_{SB}^2)} \right) \frac{(1 - \gamma)}{\theta_2} \right. \\
& - \frac{1 - \gamma}{\gamma} \sigma_r h_1 + \kappa_r \bar{r} h_1 + \frac{\sigma_X^2}{2\theta_2} \theta_1 h_2^2 + \frac{\sigma_X^2 \theta_1}{\theta_2} h_3 + \frac{\sigma_r^2}{2} h_1^2 + \sigma_X \left(\frac{\lambda_r \rho_{XB}}{\gamma} \right. \\
& + \frac{\hat{\rho}_{XS} \lambda_S}{\gamma} + \frac{\hat{\rho}_{XS} \beta_S \bar{X}}{\gamma \sigma_S \sqrt{1 - \rho_{SB}^2}} \left. \right) (1 - \gamma) h_2 - 2\sigma_r \sigma_X \rho_{XB} \theta_1 \theta_2 h_1(t) h_3(t) \left. \right] \\
& + \theta_2 r \left[h'_1 + \frac{(1 - \gamma)}{\gamma} - \kappa_r h_1 \right] \\
& + \theta_1 X \left[h'_2 + 2\kappa_X \bar{X} h_3 - \kappa_X h_2 + 2\sigma_X^2 h_3^2 - \frac{\lambda_S \beta_S}{\gamma \theta_1 \sigma_S \sqrt{1 - \rho_{SB}^2}} - \frac{\beta_S^2 \bar{X}}{\gamma \theta_1 \sigma_S^2 (1 - \rho_{SB}^2)} \right. \\
& + 2(1 - \gamma) \sigma_X h_3 \left(\frac{\lambda_r \rho_{XB}}{\gamma} + \frac{\hat{\rho}_{XS} \lambda_S}{\gamma} + \frac{\hat{\rho}_{XS} \beta_S \bar{X}}{\gamma \sigma_S \sqrt{1 - \rho_{SB}^2}} \right) - \frac{\hat{\rho}_{XS} \beta_S}{\gamma \sigma_S \sqrt{1 - \rho_{SB}^2}} (1 - \gamma) h_2 \\
& - 2\sigma_r \sigma_X \rho_{XB} \theta_1 \theta_2 h_1(t) h_3(t) \left. \right] \\
& + \theta_1 X^2 \left[h'_3 - 2\kappa_X h_3 + 2\sigma_X^2 h_3^2 + \frac{\beta_S^2}{2\gamma \sigma_S^2 (1 - \rho_{SB}^2)} \theta_1 \right. \\
& - \left. 2 \frac{\sigma_X \hat{\rho}_{XS} \beta_S (1 - \gamma)}{\gamma \sigma_S \sqrt{1 - \rho_{SB}^2}} h_3(t) \right] \tag{A.3}
\end{aligned}$$

Equation (A.3) can be written compactly as

$$\begin{aligned}
0 = & \theta_2 [h'_0 + \psi_{0,1} h_1 + \psi_{0,2} h_1^2 + \psi_{0,3} h_2 + \psi_{0,4} h_2^2 + \psi_{0,5} h_3 + \psi_{0,6}] \\
& + \theta_2 r [h'_1 + \psi_{1,1} h_1 + \psi_{1,2}] \\
& + \theta_1 X [h'_2 + \psi_{2,1} h_3 + \psi_{2,2} h_3^2 + \psi_{2,3} h_2 + \psi_{2,4} h_1(t) h_3(t) + \psi_{2,5}] \\
& + \theta_1 X^2 [h'_3 + \psi_{3,1} h_3 + \psi_{3,2} h_3^2 + \psi_{3,3}]
\end{aligned}$$

If the differential equations contained in the square brackets equal zero, then the HJB equation equals zero and $V(t, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma} \exp(h_0(t) + h_1(t)r + h_2(t)X + h_3(t)X^2)$ is a candidate for the optimal value function. \square Back to Proposition 3

Note that if the solution to the nonlinear differential equation $h'_3(t)$ exists, then the solutions to $h'_0(t)$, $h'_1(t)$, and $h'_2(t)$ also exist, because these equations are linear differential

equations. The equation $h'_1(t)$ is an ordinary differential equation which is simple to solve, and is done in Section A.2.2. An exact solution for $h'_3(t)$ is found as well. The solution for $h'_2(t)$, given the exact solutions for $h'_1(t)$ and $h'_3(t)$, do therefore also exist, see, for example Honda and Kamimura (2011).

A.2 A Verification Theorem

In this section, I will verify that the candidate solution of the HJB equation given in Equation (2.19) is the desired value function. To do so, I first state and prove a verification result for the candidate solution, before proving that the conditions in the verification theorem are fulfilled. The verification theorem, and all subsequent results, pertain to the exact solutions of the differential equations $h_i(t)$. In the following, w_u^π is the wealth evaluated in the portfolio π . The verification theorem is for the exact solutions of the equations $h_1(t)$, $h_2(t)$, and $h_3(t)$.

Proposition A.2.1 *Assume $\phi \in C^2([0, T] \times \mathbb{R} \times \mathbb{R})$, $\phi \leq 0$, $\mathcal{L}\phi \leq 0$, $\phi(T, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma}$ and $\phi(t, 0, r, X) = 0$ for each admissible control π . Then*

$$\phi(t, w, r, X) \geq \mathbb{E}\left[\frac{w_T^{1-\gamma}}{1-\gamma}\right]$$

Proof: Assume that π is an admissible control. Let $\{\tau_n\}_n$ be an increasing sequence of stopping times for the real valued process (w_u^π, r_u, X_u) , a local semimartingale, which starts in (w, r, X) at time t . Introduce the stopping times $T_n = T \wedge \tau_n$. The Itô formula gives

$$\begin{aligned} \phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n}) &= \phi(t, w, r, X) + \int_t^{T_n} \mathcal{L}\phi(u, w_u^\pi, r_u, X_u) du \\ &+ \int_t^{T_n} w \pi^\top \Sigma_t [\phi_r + \phi_w, \phi_X + \phi_w] dZ \\ &\leq \phi(t, w, r, X) + \int_t^{T_n} w \pi^\top \Sigma_t [\phi_r + \phi_w, \phi_X + \phi_w] dZ. \end{aligned}$$

Because T_n is an increasing sequence of stopping times, the Itô integral is a martingale, and hence the expectation of the integral equals zero. Taking expectation on both sides, I get

$$\phi(t, w, r, X) \geq \mathbb{E}[\phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n})]$$

Because $\phi \leq 0$, $\{\phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n})\}_n$ is a sequence of nonpositive measurable functions.

Further,

$$\lim_{n \rightarrow \infty} \phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n}) = \phi(T, w_T^\pi, r_T, s_T) = \frac{(w_T^\pi)^{1-\gamma}}{1-\gamma}.$$

The Fatou–Lebesgue lemma then implies that

$$\begin{aligned} \mathbb{E}\left[\frac{(w_T^\pi)^{1-\gamma}}{1-\gamma}\right] &= \mathbb{E}\left[\liminf_{n \rightarrow \infty} \phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n})\right] \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{E}[\phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n})] \\ &\leq \mathbb{E}\left[\limsup_{n \rightarrow \infty} \phi(T_n, w_{T_n}^\pi, r_{T_n}, X_{T_n})\right] \\ &\leq \phi(t, w, r, X) \end{aligned}$$

Which proves the proposition. \square

Proposition A.2.2 Assume as in Proposition A.2.1 that $\phi \in C^2([0, T] \times \mathbb{R}_+^3)$, $0 \geq \phi$, $\mathcal{L}\phi \leq 0$, $\phi(T, w, r, X) = \frac{w^{1-\gamma}}{1-\gamma}$ and $\phi(t, 0, r, X) = 0$ for each admissible control π . Assume furthermore that $\{\phi(\tau, w_\tau^{\pi^*}, r_\tau, X_\tau)\}_\tau$ is uniformly integrable, where π^* is an admissible control with the property that $\mathcal{L}^{\pi^*}\phi = 0$, and $t \leq \tau \leq T$ is a stopping time for the process $(w_u^{\pi^*}, r_u, X_u)$. Then

$$\phi(t, w, r, X) = V(t, w, r, X), \quad \forall (t, w, r, X) \in [0, T] \times \mathbb{R}_+^3$$

Proof: Following the structure of Proposition A.2.1, I get

$$\begin{aligned} \phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n}) &= \phi(t, w, r, X) + \int_t^{T_n} \mathcal{L}^{\pi^*} \phi(u, w_u^{\pi^*}, r_u, X_u) du \\ &\quad + \int_t^{T_n} w_u^{\pi^*} (\pi^*)^\top \Sigma_t [\phi_r(\cdot^*) + \phi_w(\cdot^*), \phi_X(\cdot^*) + \phi_w(\cdot^*)] dZ \\ &= \phi(t, w, r, X) + \int_t^{T_n} w_u^{\pi^*} (\pi^*)^\top \Sigma_t [\phi_r(\cdot^*) + \phi_w(\cdot^*), \phi_X(\cdot^*) + \phi_w(\cdot^*)] dZ \end{aligned}$$

where $\phi(\cdot^*) = \phi(u, w_u^{\pi^*}, r_u, X_u)$. Which, after taking expectations on both sides yields

$$\phi(t, w, r, X) = \mathbb{E}[\phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n})]$$

and

$$\lim_{n \rightarrow \infty} \phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n}) = \frac{(w_T^{\pi^*})^{1-\gamma}}{1-\gamma}, \quad a.s.$$

Furthermore, since the family $\{\phi(\tau, w_\tau^{\pi^*}, r_\tau, X_\tau)\}$ is uniformly integrable, I get that

$$\mathbb{E}\left[\lim_{n \rightarrow \infty} \phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n})\right] = \lim_{n \rightarrow \infty} \mathbb{E}[\phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n})]$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\frac{(w_T^{\pi^*})^{1-\gamma}}{1-\gamma}\right] &= \mathbb{E}\left[\lim_{n \rightarrow \infty} \phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n})\right] \\
&= \lim_{n \rightarrow \infty} \mathbb{E}[\phi(T_n, w_{T_n}^{\pi^*}, r_{T_n}, X_{T_n})] \\
&= \phi(t, w, r, X)
\end{aligned}$$

Which, after combining this with Proposition A.2.1, gives the desired result. \square

To prove that the candidate solution given in Equation (2.24) is in fact the optimal solution, it is necessary to prove that the function ϕ is uniformly integrable. To do this, I first make sure that the terms involving X_u^2 and their time integrals are exponentially integrable.

Lemma A.2.1 *If ξ is a constant such that*

$$\xi < \frac{\kappa_X}{2\sigma_X^2(T-t)}$$

then

$$\mathbb{E}\left[\exp\left(\xi \int_t^T X_u^2 du\right)\right] < \infty$$

Proof: Several of the theorems referred to in this proof are found in Bartle (1995). Because the expectation is obviously finite whenever $\xi \leq 0$, it remains to look at the case when $\xi > 0$. Remember that X is a Vasicek process with solution

$$X_u = e^{-\kappa_X(u-t)} X_t + \bar{X}(1 - e^{-\kappa_X(u-t)}) + \sigma_X \int_t^u e^{-\kappa_X(u-v)} dZ_X(v).$$

Further, I will make use of the fact that

$$(a+b)^2 \leq 2a^2 + 2b^2.$$

Hence

$$\begin{aligned}
\mathbb{E}[e^{\xi \int_t^T X_u^2 du}] &= \mathbb{E}[\exp\left(\xi \int_t^T \left([e^{-\kappa_X(u-t)}X + \bar{X}(1 - e^{-\kappa_X(u-t)})]du\right.\right. \\
&\quad \left.\left.+ \sigma_X \int_t^T e^{-\kappa_X(u-v)} dZ_X(v)\right)^2\right)] \\
&\leq \mathbb{E}[\exp\left(2\xi \int_t^T (e^{-\kappa_X(u-t)}X + \bar{X}(1 - e^{-\kappa_X(u-t)}))^2 du\right. \\
&\quad \left.+ 2\xi \sigma_X^2 \int_t^T Y_u^2 du\right)] \\
&= q\mathbb{E}[\exp(2\xi \sigma_X^2 \int_t^T Y_u^2 du)]
\end{aligned}$$

where

$$Y_u = \int_t^u e^{-\kappa_X(u-v)} dZ_X(v) \sim \mathcal{N}\left(0, \frac{1 - e^{-2\kappa_X(u-t)}}{2\kappa_X}\right)$$

To compute the last expectation, I express the exponential function as an infinite series, thus

$$\begin{aligned}
\mathbb{E}[\exp(2\xi \sigma_X^2 \int_t^T Y_u^2 du)] &= \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{(2\xi \sigma_X^2)^n}{n!} \left(\int_t^T Y_u^2 du\right)^n\right] \\
&= \sum_{n=0}^{\infty} \frac{(2\xi \sigma_X^2)^n}{n!} \mathbb{E}\left[\left(\int_t^T Y_u^2 du\right)^n\right] \tag{A.4}
\end{aligned}$$

where the last equality follow from the monotone convergence theorem. From Hölder's inequality, I get

$$\begin{aligned}
\int_t^T Y_u^2 du &= \int_t^T Z_u^2 du \leq \left(\int_t^T Z_u^{2n} du\right)^{\frac{1}{n}} \cdot \left(\int_t^T 1^{\frac{n}{n-1}} du\right)^{\frac{n-1}{n}} \\
&= (T-t)^{\frac{n-1}{n}} \left(\int_t^T Z_u^{2n} du\right)^{\frac{1}{n}}
\end{aligned}$$

which implies that Equation (A.4) becomes

$$\sum_{n=0}^{\infty} \frac{(2\xi \sigma_X^2)^n}{n!} \mathbb{E}\left[\left(\int_t^T Y_u^2 du\right)^n\right] \leq \sum_{n=0}^{\infty} \frac{(2\xi \sigma_X^2)^n}{n!} \mathbb{E}\left[\left(\int_t^T Y_u^{2n} du\right)\right]$$

Because all continuous functions are measurable, and Y_u^{2n} is continuous, Tonelli's Theorem yields

$$\mathbb{E}\left[\int_t^T Y_u^{2n} du\right] = \int_t^T \mathbb{E}[Y_u^{2n} du]$$

Hence

$$\sum_{n=0}^{\infty} \frac{(2\xi\sigma_X^2)^n}{n!} (T-t)^{\frac{n-1}{n}} \mathbb{E} \left[\int_t^T Z_u^{2n} du \right] = (T-t)^{-1} \sum_{n=0}^{\infty} \frac{(2\xi\sigma_X^2(T-t))^n}{n!} \int_t^T \mathbb{E}[Z_u^{2n}] du$$

Because $Z_u \sim \mathcal{N}(0, \gamma_u^2)$, I find that

$$\begin{aligned} \mathbb{E}[Z_u^{2n}] &= \frac{1}{\sqrt{2\pi\gamma_u^2}} \int_{-\infty}^{\infty} z^{2n} e^{-\frac{z^2}{2\gamma_u^2}} dz \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{1}{2\gamma_u^2} \right)^{\frac{1}{2}} \frac{1}{2} \left(\frac{1}{2\gamma_u^2} \right)^{-\frac{2n+1}{2}} \Gamma\left(\frac{2n+1}{2}\right) \\ &= \frac{1}{\sqrt{\pi}} (2\gamma_u^2)^n \Gamma\left(\frac{2n+1}{2}\right) \end{aligned}$$

Further, since $\kappa_X > 0$,

$$\gamma_u^2 = \frac{1}{2\kappa_X} \left(1 - e^{-2\kappa_X(u-t)} \right) \leq \frac{1}{2\kappa_X}.$$

Which gives

$$\mathbb{E}[Z_u^{2n}] = \frac{1}{\sqrt{\pi}} (2\gamma_u^2)^n \Gamma\left(\frac{2n+1}{2}\right) \leq \frac{1}{\sqrt{\pi}} \kappa_X^{-n} \Gamma\left(\frac{2n+1}{2}\right)$$

Hence

$$\begin{aligned} & (T-t)^{-1} \sum_{n=0}^{\infty} \frac{(2\sigma_X^2 \xi(T-t))^n}{n!} \int_t^T \mathbb{E}[Z_u^{2n}] du \\ & \leq (T-t)^{-1} \sum_{n=0}^{\infty} \frac{(2\sigma_X^2 \xi(T-t))^n}{n!} \int_t^T \mathbb{E} \left[\frac{1}{\sqrt{\pi}} \kappa_X^{-n} \Gamma\left(\frac{2n+1}{2}\right) \right] du \\ & = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2\sigma_X^2 \xi(T-t) \kappa_X^{-1})^n}{n!} \Gamma\left(\frac{2n+1}{2}\right) \end{aligned}$$

From the ratio test, this series is infinite whenever

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(2\sigma_X^2 \xi(T-t) \kappa_X^{-1})^{n+1}}{(n+1)!} \Gamma\left(\frac{2n+3}{2}\right)}{\frac{(2\sigma_X^2 \xi(T-t) \kappa_X^{-1})^n}{n!} \Gamma\left(\frac{2n+1}{2}\right)} &= \lim_{n \rightarrow \infty} \frac{1}{n+1} (2\sigma_X^2 \xi(T-t) \kappa_X^{-1}) (n + \frac{1}{2}) \\ &= 2\sigma_X^2 \xi(T-t) \kappa_X^{-1} < 1 \end{aligned}$$

which holds since $\xi < \frac{\kappa}{2\sigma_X^2(T-t)}$ and the proof is complete. \square

Lemma A.2.2 $Z_{t_1} := e^{k(\int_t^{t_1} e^{\kappa_X u} dZ_X(u))^2} \in L^1(P)$ for all $t \leq t_1 \leq T$ when $k < \frac{\kappa_X}{\exp(2\kappa_X T) - \exp(2\kappa_X t)}$.

Proof: Because $Z_{t_1} > 0$ it is sufficient to prove that $\mathbb{E}[Z_{t_1}] < \infty$. This inequality follows

directly if $k \leq 0$, so it remains to check the case when $k > 0$.

$$\begin{aligned}\mathbb{E}[Z_{t_1}] &= \mathbb{E}[\exp(k(\int_t^{t_1} e^{\kappa_X u} dZ_X(u))^2)] \\ &= \mathbb{E}[\sum_{n=0}^{\infty} \frac{k^n}{n!} (\int_t^{t_1} e^{\kappa_X u} dZ_X(u))^{2n}] \\ &= \sum_{n=0}^{\infty} \frac{k^n}{n!} \mathbb{E}[w_{t_1}^{2n}]\end{aligned}$$

where

$$w_{t_1} = \int_t^{t_1} e^{\kappa_X u} dZ_X(u)$$

so

$$w_{t_1} \sim \mathcal{N}(0, \frac{1}{2\kappa_X}(e^{2\kappa_X t_1} - e^{2\kappa_X t}))$$

Because $w_{t_1} \sim \mathcal{N}(0, \gamma_{t_1}^2)$,

$$\begin{aligned}\mathbb{E}[w_{t_1}^{2n}] &= \frac{1}{\sqrt{2\pi\gamma_{t_1}^2}} \int_{-\infty}^{\infty} w^{2n} e^{-\frac{w^2}{2\gamma_{t_1}^2}} dw \\ &= \frac{1}{\sqrt{\pi}} (2\gamma_{t_1}^2)^n \Gamma(n + \frac{1}{2})\end{aligned}$$

Inserting this, and the expression for $\gamma_{t_1}^2$, into the formula for $\mathbb{E}[Z_{t_1}]$ we get

$$\mathbb{E}[Z_{t_1}] = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{k^n}{n!} \left(\frac{e^{2\kappa_X t_1} - e^{2\kappa t}}{\kappa_X} \right)^n \Gamma(n + \frac{1}{2})$$

From the ratio test, this series converges whenever $k < \frac{\kappa_X}{e^{2\kappa_X t_1} e^{2\kappa t}}$, and because this must hold for all $t \leq t_1 \leq T$, this holds and the proof is complete. \square

A.2.1 Solution of the Control Problem

In this section, I will show that the candidate solution is the optimal value function, that is $\phi(t, w, r, X) = V(t, w, r, s)$. This means that I have to show that $\phi(t, w, r, X)$ satisfies the conditions in Proposition A.2.2. The wealth dynamics, after substituting the expressions

for the optimal trading policies π_B^* and π_S^* , can be written as

$$\begin{aligned}
dw_t^* &= \left[\left(\frac{\sigma_r \lambda_r \beta_S \rho_{SB} \bar{X} + \sigma_r \lambda_r^2 \sigma_S}{\gamma \sigma_r \sigma_S (1 - \rho_{SB}^2)} \sigma_r \lambda_r h_1(t) - \frac{\sigma_r \lambda_r \theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma \sigma_r (1 - \rho_{SB}^2)} h_2(t) + \frac{\beta_S^2 \bar{X}^2}{\gamma \sigma_S^2 (1 - \rho_{SB}^2)} \right. \right. \\
&\quad \left. \left. - \frac{\theta_1 \sigma_X \beta_S \rho_{SX} \bar{X}}{\gamma \sigma_S (1 - \rho_{SB}^2)} h_2(t) \right) + r + \left(\frac{\theta_1 \sigma_X \beta_S \rho_{SX}}{\gamma \sigma_S (1 - \rho_{SB}^2)} h_2(t) - \frac{2\beta_S^2 \bar{X}}{\gamma \sigma_S^2 (1 - \rho_{SB}^2)} \right. \right. \\
&\quad \left. \left. - \frac{2\theta_1 \sigma_X \beta_S \rho_{SX} \bar{X}}{\gamma \sigma_S (1 - \rho_{SB}^2)} h_3(t) - \frac{\lambda_r \rho_{SB} \beta_S}{\gamma \sigma_S (1 - \rho_{SB}^2)} - \frac{2\theta_1 \lambda \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} h_3(t) \right) X \right. \\
&\quad \left. + \left(\frac{\beta_S^2}{\gamma \sigma_S^2 (1 - \rho_{SB}^2)} + \frac{2\theta_1 \sigma_X \beta_S \rho_{SX}}{\gamma \sigma_S (1 - \rho_{SB}^2)} h_3(t) \right) X^2 \right] w dt \\
&\quad + \left[\left(\frac{\beta_S \bar{X} \rho_{SB} + \lambda \sigma_S}{\gamma \sigma_S (1 - \rho_{SB}^2)} - \frac{\theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} h_2(t) + \sigma h_1(t) \right) - X \left(\frac{\beta_S \rho_{SB}}{\gamma \sigma_S (1 - \rho_{SB}^2)} \right. \right. \\
&\quad \left. \left. + \frac{2\theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} h_3(t) \right) \right] w dZ_r + \left[\left(\frac{\theta_1 \sigma_X \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} \right. \right. \\
&\quad \left. \left. - \frac{\beta_S \bar{X} + \lambda \sigma_S \rho_{SB}}{\gamma \sigma_S (1 - \rho_{SB}^2)} \right) + \left(\frac{\beta_S}{\gamma \sigma_S (1 - \rho_{SB}^2)} + \frac{2\theta_1 \sigma_X \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} h_3(t) \right) X \right] w dZ_S \\
&= (\alpha_0(t) + \alpha_1(t)r + \alpha_2(t)X + \alpha_3(t)X^2) w dt + (\sigma_1(t) + \sigma_2(t)X) w dZ_r \\
&\quad + (\sigma_3(t) + \sigma_4(t)X) w dZ_S
\end{aligned}$$

where the solution is given by

$$\begin{aligned}
w_s^* &= w \exp \left(\int_t^s \left((\alpha_0 - \left(\frac{\sigma_1^2 + \sigma_3^2 + 2\rho_{SB} \sigma_1 \sigma_3}{2} \right)) + \alpha_1 r + (\alpha_2 - \sigma_1 \sigma_2 - \sigma_3 \sigma_4 \right. \right. \\
&\quad \left. \left. - \rho_{SB} (\sigma_1 \sigma_4 + \sigma_2 \sigma_3)) X + (\alpha_3 - \frac{\sigma_2^2 + \sigma_4^2 + 2\rho_{SB} \sigma_2 \sigma_4}{2}) X^2 \right) du \right. \\
&\quad \left. + \int_t^s (\sigma_1 + \sigma_2 X) dZ_r(u) + \int_t^s (\sigma_3 + \sigma_4 X) dZ_S(u) \right)
\end{aligned}$$

Lemma A.2.3 π^* is an admissible control.

Proof: The interest rate, r_t , and the predictor, X_t , are both continuous functions hence π^* consists of continuous functions and the integrability condition and the adapted condition follows. That the solution of w_{t_1} is positive is obvious, so it remains to prove that the solution is unique. Using the same notation as in Theorem 5.2.1 in Øksendal (2003), I get the following

$$\begin{aligned}
b(t, w, r, X) &= (\alpha_0(t) + \alpha_1(t)r + \alpha_2(t)X + \alpha_3(t)X^2)w \\
\sigma_i(t, w, r, X) &= (\sigma_1(t) + \sigma_2(t)X)w \\
\sigma_{ii}(t, w, r, X) &= (\sigma_3(t) + \sigma_4(t)X)w
\end{aligned}$$

Because all these functions are finite, condition 5.2.2 in Øksendal (2003) is satisfied,¹ and hence the solution is unique. \square

That $\mathcal{L}^{\pi^*} \phi = 0$ is proven in Proposition 2.2.3. The next proposition gives conditions on the parameters of the control problem which guaranties uniform integrability.

Proposition A.2.3 *Assume that there exists two positive constants ϵ and δ such that*

$$(2(1 + \epsilon)h_3(t) + \delta)(\delta + \sigma_X^2 e^{-2\kappa t}) < \frac{\kappa_X}{e^{2\kappa_X T} - e^{2\kappa_X t}} \quad (\text{A.5})$$

$$8(1 + \epsilon)(1 - \gamma)\left(\delta\sigma_X^2\gamma + \frac{\beta_S^2}{\sigma_S^2(1 - \rho_{SB}^2)}\right) < \frac{\kappa_X\gamma}{T - t} \quad (\text{A.6})$$

$$32(1 + \epsilon)^2(1 - \gamma)^2\left(\delta + \left(\frac{\beta_S\hat{\rho}}{\gamma\sigma_S(1 - \rho_{SB}^2)}\right)^2 + \left(\frac{2\theta_1\sigma_X\hat{\rho}\rho_{SX}}{\gamma(1 - \rho_{SB}^2)}\right)^2 h_3(t)^2\right) < \frac{\kappa_X}{2\sigma_X^2(T - t)} \quad (\text{A.7})$$

Then the family $\{\phi(\tau, w_\tau^*, r_\tau, X_\tau)\}$ is uniformly integrable.

Proof: From Theorem C.4. in Øksendal (2003) the proof of uniform integrability is equivalent to prove as to prove that

$$\sup_{t \leq \tau \leq T} \mathbb{E}[\phi^{1+\epsilon}(\tau, w_\tau^*, r_\tau, X_\tau)] < \infty$$

Let $G(t, r, X) = f(t, X)^{\theta_1} g(t, r)^{\theta_2}$, as in Equation (A.1). By the Cauchy-Schwartz inequality, I get

$$\begin{aligned} \mathbb{E}[\phi^{1+\epsilon}(\tau, w_\tau^*, r_\tau, X_\tau)] &= \mathbb{E}\left[\left(\frac{(w_\tau^*)^{1-\gamma}}{1-\gamma}\right)^{1+\epsilon} G(t, r, X)^{1+\epsilon}\right] \\ &\leq (1 - \gamma)^{-(1+\epsilon)} \mathbb{E}\left[(w_\tau^*)^{2(1-\gamma)(1+\epsilon)}\right]^{\frac{1}{2}} \mathbb{E}\left[G(t, r, X)^{2(1+\epsilon)}\right]^{\frac{1}{2}} \\ &\leq \underbrace{\frac{\mathbb{E}\left[(w_\tau^*)^{2(1-\gamma)(1+\epsilon)}\right]^{\frac{1}{2}}}{(1 - \gamma)^{(1+\epsilon)}}}_1 \underbrace{\mathbb{E}\left[f(t, s)^{2\theta_1(1+\epsilon)}\right]^{\frac{1}{2}}}_2 \\ &\quad \underbrace{\mathbb{E}\left[g(t, r)^{2\theta_2(1+\epsilon)}\right]^{\frac{1}{2}}}_3 \end{aligned} \quad (\text{A.8})$$

I have to show that all three terms are finite. I start with term 2 : Because $h_2(t)$ is bounded,

¹Condition 5.2.2 in Øksendal (2003) is a Lipschitz condition in that it states that for a function $f(t, \cdot) = b(t, \cdot) + \sigma(t, \cdot)$, the following holds for a constant D : $|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|$.

a constant δ implies that

$$0 \leq \left(\frac{(1+\epsilon)h_2(\tau)}{\sqrt{\delta}} - \sqrt{\delta}X_\tau \right)^2 = \frac{(1+\epsilon)^2 h_2(\tau)^2}{\delta} - 2(1+\epsilon)h_2(\tau)X_\tau + \delta X_\tau^2$$

which again gives

$$2(1+\epsilon)h_2(\tau)X_\tau \leq \frac{(1+\epsilon)^2 h_2(\tau)^2}{\delta} + \delta X_\tau^2 = \zeta + \delta X_\tau^2$$

Where ζ is a constant. In the following, ζ will always be a constant, but different from case to case. The important issue is the boundedness of the term in question, being either bounded below or above. Now, define $\xi := \sup_{t \leq \tau \leq T} (2(1+\epsilon)h_3(\tau) + \delta)$. From this I get that

$$\begin{aligned} \mathbb{E} \left[f(t, s)^{2\theta_1(1+\epsilon)} \right]^{\frac{1}{2}} &= \mathbb{E} \left[\exp \left(2(1+\epsilon)h_2(\tau)X_\tau + 2(1+\epsilon)h_3(\tau)X_\tau^2 \right) \right]^{\frac{1}{2}} \\ &\leq \zeta \mathbb{E} \left[\exp \left((2(1+\epsilon)h_3(\tau) + \delta)X_\tau^2 \right) \right]^{\frac{1}{2}} \\ &\leq \zeta \mathbb{E} \left[\exp (\xi X_\tau^2) \right]^{\frac{1}{2}} \end{aligned}$$

Recall the expression of X_τ , which is given as

$$X_\tau = X_0 e^{-\kappa_X \tau} + \bar{X}(1 - e^{-\kappa_X \tau}) + \int_t^\tau \sigma_X e^{\kappa_X(s-t)} dZ_s$$

Further observe that, since

$$0 \leq \left(\frac{\sigma_X e^{-\kappa_X(2\tau-t)}(s + \bar{X}e^{-\kappa_X t})}{\sqrt{\delta}} - \sqrt{\delta} \int_t^\tau e^{\kappa_X u} dZ_X(u) \right)^2$$

the following inequality holds

$$2\sigma_X e^{-\kappa_X(2\tau-t)}(X + \bar{X}e^{-\kappa_X \tau}) \int_t^\tau e^{\kappa_X u} dZ_X(u) \leq \zeta + \delta \left(\int_t^\tau e^{\kappa_X u} dZ_X(u) \right)^2$$

where ζ is a constant larger than $\frac{\sigma_X^2 e^{-2\kappa_X(2\tau-t)}(s + \bar{X}e^{-\kappa_X t})^2}{\delta}$, thus different than the preced-

ing. This implies that, when $X = X_t$:

$$\begin{aligned}
\mathbb{E}\left[e^{\xi X_\tau^2}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\exp\left(\xi\left(\frac{X^2}{e^{2\kappa_X(\tau-t)}} + \bar{X}^2(1 - e^{-\kappa_X(\tau-t)})^2 \frac{\sigma_X^2}{e^{2\kappa_X\tau}}\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right.\right.\right. \\
&\quad + \left.2e^{-\kappa_X(\tau-t)}X\bar{X}(1 - e^{-\kappa_X(\tau-t)})\right. \\
&\quad \left.+ \left.2\sigma_X e^{-\kappa_X\tau}(\bar{X}(1 - e^{-\kappa_X(\tau-t)}) + X e^{-\kappa_X(\tau-t)})\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)\right]^{\frac{1}{2}} \\
&\leq \zeta \mathbb{E}\left[\exp\left(\xi(\sigma_X^2 e^{-2\kappa_X\tau}\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right.\right. \\
&\quad \left.+ \left.2\sigma_X e^{-\kappa_X(2\tau-t)}(X + \bar{X} e^{-\kappa_X\tau})\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)\right]^{\frac{1}{2}}
\end{aligned}$$

Which again implies that

$$\mathbb{E}\left[f(t, X)^{2(1+\epsilon)\theta_1}\right]^{\frac{1}{2}} \leq \zeta \mathbb{E}\left[\exp(\xi(\delta + \sigma_X^2 e^{-2\kappa_X t})\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2)\right]^{\frac{1}{2}}$$

Because $\int_t^s e^{\kappa_X u} dZ_X(u)$ is a martingale, Jensen's inequality ensures that for any positive constant k ,

$$X_s := \exp(k\left(\int_t^s e^{\kappa_X u} dZ_X(u)\right)^2)$$

is a positive submartingale, provided that $X_s \in L^1(P)$ for all $s \in (t, T)$. If now

$$k = \xi(\delta + \sigma_X^2 e^{-2\kappa_X t})$$

then Lemma A.2.2 is satisfied by condition (A.5). This means that I can apply Doob's martingale inequality. Letting $p = 1 + \delta$ and $q = 1 + \frac{1}{\delta}$:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(k\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right)\right] &= \left\|\exp\left(\frac{1}{p}\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right)\right\|_p^p \\
&\leq \left\|\sup_{t \leq s \leq T} \exp\left(\frac{1}{p}\left(\int_t^s e^{\kappa_X u} dZ_X(u)\right)^2\right)\right\|_p^p \\
&\stackrel{\text{Doob}}{\leq} q^p \sup_{t \leq s \leq T} \left\|\exp\left(k\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right)\right\|_p^p \\
&= q^p \sup_{t \leq s \leq T} \mathbb{E}\left[\exp\left(k\left(\int_t^\tau e^{\kappa_X u} dZ_X(u)\right)^2\right)\right] \\
&\leq \zeta
\end{aligned}$$

Thus it is proved that

$$\mathbb{E}^t\left[f(t, X)^{2\theta_2(1+\epsilon)}\right]^{\frac{1}{2}} \leq \zeta$$

I will now continue by proving term 3 in Equation (A.8). Because the function $g(t, r) =$

$\exp(h_0(t) + h_1(t)r)$, the following holds

$$g(t, r)^{2\theta_2(1+\epsilon)} = \exp(2\theta_2(1+\epsilon)(h_0(t) + h_1(t)r)).$$

Because r is normally distributed, the expectation of this expression is given by

$$\begin{aligned} & \mathbb{E}\left[g(t, r)^{2\theta_2(1+\epsilon)}\right] \\ = & \mathbb{E}\left[\exp(2\theta_2(1+\epsilon)(h_0(t) + h_1(t)r))\right] \\ = & \exp\left(\mathbb{E}\left[2\theta_2(1+\epsilon)(h_0(t) + h_1(t)r)\right] + \frac{1}{2}\text{Var}\left[2\theta_2(1+\epsilon)(h_0(t) + h_1(t)r)\right]\right) \\ = & \exp\left(2\theta_2(1+\epsilon)(h_0(t) + h_1(t)\mathbb{E}[r]) + 2(\theta_2(1+\epsilon)h_1(t))^2\text{Var}[r]\right) \\ = & \exp\left(2\gamma(1+\epsilon)[h_0(t) + h_1(t)m + h_1(t)(r_0 - m)e^{-\alpha t}\right. \\ & \left. + \gamma^2(1+\epsilon)^2\sigma^2h_1^2(t)(1 - e^{-2\alpha t})\right] \\ < & \infty \end{aligned}$$

The last inequality follows from the fact that all $h_i(t)$ are bounded. The expectation for term 3 in Equation (A.8) is therefore satisfied. I will now prove that

$$\mathbb{E}^t\left[(w_\tau^*)^{2(1-\gamma)(1+\epsilon)}\right]^{\frac{1}{2}} \leq \zeta$$

as given in term 1 above. Let $\varepsilon = (1 - \gamma)(1 + \epsilon)$. By the Cauchy-Schwartz inequality, the following holds

$$\begin{aligned} & \mathbb{E}^t\left[(w_\tau^*)^{2\varepsilon}\right] \\ \leq & \zeta \mathbb{E}^t\left[\exp\left(2\varepsilon \int_t^\tau \left(\alpha_1 r + (\alpha_2 - \sigma_1\sigma_2 - \sigma_3\sigma_4 - \rho_{SB}(\sigma_1\sigma_4 + \sigma_2\sigma_3))X\right.\right.\right. \\ & \left. + \left.\left(\alpha_3 - \frac{\sigma_2^2 + \sigma_4^2 + 2\rho_{SB}\sigma_2\sigma_4}{2}\right)X^2\right)du + 2\varepsilon\left[\int_t^s (\sigma_1 + \sigma_2 X)dZ_r(u)\right.\right. \\ & \left. + \left.\int_t^s (\sigma_3 + \sigma_4 X)dZ_S(u)\right]\right] \\ = & \mathbb{E}^t\left[\exp\left(2\varepsilon\left(\int_t^\tau Fdu + \int_t^\tau GdZ_r(u) + \int_t^\tau HdZ_S(u)\right)\right)\right] \\ \stackrel{C.S}{\leq} & \zeta \mathbb{E}^t\left[\exp\left(4\varepsilon \int_t^\tau Fdu\right)\right]^{\frac{1}{2}} \times \mathbb{E}^t\left[\exp\left(4\varepsilon \int_t^\tau GdZ_r(u)\right)\right]^{\frac{1}{2}} \\ \times & \mathbb{E}^t\left[\exp\left(4\varepsilon \int_t^\tau HdZ_S(u)\right)\right]^{\frac{1}{2}} \end{aligned}$$

I proceed by proving that all three expectations are finite. First, I observe that

$$\begin{aligned}
& (\alpha_2 - \sigma_1\sigma_2 - \sigma_3\sigma_4 - \rho_{SB}(\sigma_1\sigma_4 + \sigma_2\sigma_3))X \\
& \leq \frac{(\alpha_2 - \sigma_1\sigma_2 - \sigma_3\sigma_4 - \rho_{SB}(\sigma_1\sigma_4 + \sigma_2\sigma_3))^2}{4\delta} + \delta X^2 \\
& \leq \zeta + \delta X^2
\end{aligned}$$

Moreover, since $h_3(t) \geq 0$ and reasonable estimates of the correlation between the stock returns and bonds are positive, $\rho_{SB} > 0$, and the correlation between the stock returns and the predictable variable indicate that $\rho_{SX} < 0$, the following inequality holds

$$\begin{aligned}
\alpha_3(t) - \frac{\sigma_1^2 + \sigma_4^2 + 2\rho_{SB}\sigma_2\sigma_4}{2} & \leq \alpha_3(t) \\
& = \frac{\beta_S^2}{\gamma\sigma_S^2(1 - \rho_{SB}^2)} + \frac{2\theta_1\beta_S\sigma_X\rho_{SX}}{\gamma\sigma_S(1 - \rho_{SB}^2)}h_3(t) \\
& \leq \frac{\beta_S^2}{\gamma\sigma_S^2(1 - \rho_{SB}^2)}
\end{aligned}$$

Because $\alpha_1 = 1$, the solution of the differential equation describing the interest rate r_t makes the integral of r_t bounded. Hence, the following inequality holds for the first expectation

$$\mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^\tau F du \right) \right]^{\frac{1}{2}} \leq \zeta \mathbb{E}^t \left[2\varepsilon \left(\int_t^\tau r_u du + \left(\delta + \frac{\beta_S^2}{\gamma\sigma_S^2(1 - \rho_{SB}^2)} \right) \int_t^\tau X_u^2 du \right) \right].$$

The expectation on the right hand side is finite by Lemma A.2.1 whenever Equation (A.6) holds. Now, by applying Doob's inequality on the second expectation, I get

$$\begin{aligned}
\mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^\tau G dZ_r(u) \right) \right]^{\frac{1}{2}} & = \left\| \exp \left(2\varepsilon \int_t^\tau G dZ_r(u) \right) \right\|_{L_2} \\
& = \left\| \sup_{t \leq s \leq T} \exp \left(2\varepsilon \int_t^s G dZ_r(u) \right) \right\|_{L_2} \\
& \stackrel{\text{Doob}}{\leq} 2 \sup_{t \leq s \leq T} \left\| \exp \left(2\varepsilon \int_t^s G dZ_r(u) \right) \right\|_{L_2} \\
& = 2 \sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) \right) \right]^{\frac{1}{2}}
\end{aligned}$$

This holds if $X_s = \exp(2\varepsilon \int_t^s G dZ_r(u))$ is a positive submartingale, which is the case

according to Jensen's inequality provided that $X_s \in L^1(P)$. Moreover

$$2 \sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) \right) \right]^{\frac{1}{2}} \stackrel{*}{\leq} 2 \mathbb{E}^t \left[\exp \left(32\varepsilon^2 \int_t^s G^2 du \right) \right]^{\frac{1}{4}}$$

In (*), I applied the following argument

$$\begin{aligned} & \sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) \right) \right]^{\frac{1}{2}} \\ &= \sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) - 16\varepsilon^2 \int_t^s G^2 du + 16\varepsilon^2 \int_t^s G^2 du \right) \right]^{\frac{1}{2}} \\ &\leq \sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(8\varepsilon \int_t^s G dZ_r(u) \right. \right. \\ &\quad \left. \left. - 32\varepsilon^2 \int_t^s G^2 du \right) \right]^{\frac{1}{4}} \mathbb{E}^t \left[\exp \left(32\varepsilon^2 \int_t^s G^2 du \right) \right]^{\frac{1}{4}} \end{aligned}$$

Thus, if

$$X_s = \exp \left(8\varepsilon \int_t^s G dZ_r(u) - 32\varepsilon^2 \int_t^s G^2 du \right)$$

is a martingale, then $\mathbb{E}^t[X_s] = 1$ for $t \leq s \leq T$. This fact will be proven later. Because

$$\sup_{t \leq s \leq T} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) \right) \right]^{\frac{1}{4}} = \mathbb{E}^t \left[\exp \left(32\varepsilon^2 \int_t^T G^2 du \right) \right]^{\frac{1}{4}}$$

the inequality in (*) follows. Further,

$$\mathbb{E}^t \left[\exp \left(32\varepsilon^2 \int_t^T G^2 du \right) \right] \leq \zeta \mathbb{E}^t \left[\exp \left(32\varepsilon^2 \int_t^T (\delta + \sigma_2(t)^2) s^2 du \right) \right]$$

Moreover, since $\rho_{SX} < 0$ and $h_3(t) \geq 0$, I get that

$$\begin{aligned} \sigma_2^2(t) &= \left(\frac{\beta_S \rho_{SB}}{\gamma \sigma_S (1 - \rho_{SB}^2)} + \frac{2\theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} h_3(t) \right)^2 \\ &\leq \left(\frac{\beta_S \rho_{SB}}{\gamma \sigma_S (1 - \rho_{SB}^2)} \right)^2 + \left(\frac{2\theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} \right)^2 \sup_{t \leq u \leq T} h_3^2(u) \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}^t \left[\exp \left(4\varepsilon \int_t^s G dZ_r(u) \right) \right]^{\frac{1}{2}} &\leq \zeta \mathbb{E}^t \left[\exp \left(32\varepsilon^2 \left(\delta + \left(\frac{\beta_S \rho_{SB}}{\gamma \sigma_S (1 - \rho_{SB}^2)} \right)^2 \right. \right. \right. \\ &\quad \left. \left. \left. + \left(\frac{2\theta_1 \sigma_X \rho_{SB} \rho_{SX}}{\gamma (1 - \rho_{SB}^2)} \right)^2 h_3^2(T) \right) \int_t^T X_u^2 du \right) \right]^{\frac{1}{4}} \end{aligned}$$

This is finite by Lemma A.2.1 whenever the condition in Equation (A.7) holds with $\hat{\rho} = \rho_{SB}$. Notice that this implies that

$$2\mathbb{E}\left[\exp\left(32\varepsilon^2 \int_t^T G^2 du\right)\right]^{\frac{1}{4}} < \infty$$

which completes the proof of (*). The third expectation is proved in a similar fashion, with $\hat{\rho} = 1$. Thus, I have proved that

$$\mathbb{E}\left[\left(w_\tau^*\right)^{2(1-\gamma)(1+\epsilon)}\right] < \infty \quad (\text{A.9})$$

and I conclude that

$$\mathbb{E}\left[\phi^{1+\epsilon}(\tau, w_\tau^*, r_\tau, X_\tau)\right] < \infty \quad (\text{A.10})$$

with respect to all stopping times $t \leq \tau \leq T$. This means that

$$\sup_{t \leq \tau \leq T} \mathbb{E}^t\left[\phi^{1+\epsilon}(\tau, w_\tau^*, r_\tau, X_\tau)\right] < \infty$$

and hence the family $\{\phi^{1+\epsilon}(\tau, w_\tau^*, r_\tau, X_\tau)\}_\tau$ is uniformly integrable. \square

A.2.2 Solution of the functions $h_i(t)$

In the optimal trading policies there are several deterministic functions, represented by $h_1(t)$, $h_2(t)$, and $h_3(t)$. Let τ be the time left of the investment horizon, so $\tau = T - t$. Note that for the results of Proposition 2.2.3 to be valid, the existence of the solution for the differential equations is required and not necessarily specific analytical solutions, see for example Honda and Kamimura (2011) for a verification theorem of the portfolio problem represented in Kim and Omberg (1996). However, to represent the optimal asset allocation with tables and figures, I will in this section take a closer look at how to solve these functions explicitly, or approximately. The function $h_0(t)$ is not present in the optimal trading policies, so I will not give a solution to this function. All the functions $h_i(t)$ converge to zero in time. Polyanin and Zaitsev (2003) give exact solutions to the differential equations described in this section. A differential equation of the form

$$\begin{aligned} \frac{\partial}{\partial t} f(t) + b f(t) + c &= 0 \\ f(T) &= 0 \end{aligned} \quad (\text{A.11})$$

has the solution

$$f(t) = \frac{c}{b}(\exp(b \cdot \tau) - 1). \quad (\text{A.12})$$

Function $h_1'(t)$

The function $h_1(t)$ is of the form of Equation (A.12), with $b = \psi_{1,1} = -\kappa_r$ and $c = \psi_{1,2} = \frac{1-\gamma}{\gamma}$. The solution is therefore

$$h_1 = \frac{1-\gamma}{\kappa_r \gamma} \left(1 - \exp(-\kappa_r \cdot \tau) \right). \quad (\text{A.13})$$

Function $h_3'(t)$

Because the ODE $h_2'(t)$ include the function $h_3(t)$, I will continue by giving the solution of the ordinary differential equation (ODE) for $h_3'(t)$. The solution to the first-order nonlinear ordinary differential equation

$$\begin{aligned} \frac{\partial}{\partial t} h_3(t) + \psi_{3,1} h_3(t) + \psi_{3,2} h_3(t)^2 + \psi_{3,3} &= 0. \\ h_3(T) &= 0 \end{aligned} \quad (\text{A.14})$$

has four distinct forms. When $\psi_{3,1} < 0$, $\psi_{3,2} > 0$, and $\psi_{3,3} > 0$, as is the case for parameters applied in this paper, the solution is

$$\begin{aligned} h_3(t) &= \frac{1}{2\psi_{3,2}} \left[\psi_{3,1} + \tanh \left(\frac{\tau}{2} \sqrt{4\psi_{3,2}\psi_{3,3} - \psi_{3,1}^2} \right. \right. \\ &\quad \left. \left. + \tanh^{-1} \left(\frac{-\psi_{3,1}}{\sqrt{4\psi_{3,2}\psi_{3,3} - \psi_{3,1}^2}} \right) \right) \sqrt{4\psi_{3,2}\psi_{3,3} - \psi_{3,1}^2} \right] \end{aligned} \quad (\text{A.15})$$

For the function $h_3(t)$ to be well-defined, the absolute value of the argument of $\tanh^{-1}(\cdot)$ must be less than one. This is the case, as $\sqrt{4\psi_{3,2}\psi_{3,3} - \psi_{3,1}^2} > -\psi_{3,1}$.² The other three solutions are similar to this, but the expressions under the square root changes, depending on the sign of the parameters $\psi_{3,1}$, $\psi_{3,2}$ and $\psi_{3,3}$. However, to be able to solve the ODE $h_2'(t)$, which include the solution of $h_3'(t)$, it is necessary to simplify the ODE $h_3'(t)$ slightly and write it of the form given in Equation (A.11). This approximation is carried out by assuming that the nonlinear term $\psi_{3,2} h_3^2(t) = 2\sigma_X^2 h_3(t)$ is equal to zero. The solution of

²The argument of the square root is positive. If the parameters resulted in a negative value for $4 \cdot \psi_{3,2} \cdot \psi_{3,3} - \psi_{3,1}^2$, then the expression under the square root would change to become positive, for example $\psi_{3,1}^2 - 4 \cdot \psi_{3,2} \cdot \psi_{3,3}$, see Polyanin and Zaitsev (2003).

the simplified ODE $h'_3(t)$ is

$$h_3^a(t) = \frac{\psi_{3,3}}{\psi_{3,1}}(1 - \exp(\psi_{3,1} \cdot \tau)). \quad (\text{A.16})$$

The superscript a refer to *approximation*. The coefficients $\psi_{3,1}$ and $\psi_{3,3}$ are given by

$$\begin{aligned} \psi_{3,1} &= 2\beta_S \left[\kappa_X - \frac{\sigma_X \rho_{SX}(1 - \gamma)}{\gamma \sigma_S(1 - \rho_{SB}^2)} \right] \\ \psi_{3,3} &= \frac{\beta_S^2}{\gamma \sigma_S^2(1 - \rho_{SB}^2)}. \end{aligned}$$

An important question is then: *how good is this approximation?* The relative difference between $h_3(t)$ and $h_3^a(t)$ for all $t \in [0, T]$, is then

$$h_3^d(t) = \frac{h_3(t) - h_3^a(t)}{h_3^a(t)}. \quad (\text{A.17})$$

For parameters applied in this paper, the function $h_3^d(t)$ is largest at $t = 0$, with a value of 0.004, or a 0.4 percent difference between the functions. The two solutions of $h'_3(t)$ therefore yields very similar results under the parameter estimates of this paper.³

Function $h'_2(t)$

The differential equation is given by, for $\tau = \cdot \tau$:

$$\begin{aligned} 0 &= h'_2(t) + \psi_{2,1}h_2(t) + \psi_{2,2}h_3^2(t) + \psi_{2,3}h_3(t) + \psi_{2,4}h_1(t)h_3(t) + \psi_{2,5} \\ &= h'_2(t) + \psi_{2,1}h_2(t) + \Psi_{2,3}(e^{-\psi_{3,1}\tau} - 1) + \Psi_{2,4}(e^{-\psi_{1,1}\tau} - 1)(1 - e^{-\psi_{3,1}\tau}) + \psi_{2,5} \end{aligned}$$

Because $\psi_{2,2} = \psi_{3,2}$, the coefficients $\Psi_{2,3}$ and $\Psi_{2,4}$ are given by

$$\begin{aligned} \Psi_{2,3} &= \frac{1}{4\psi_{3,2}} \\ \Psi_{2,4} &= -\frac{\psi_{2,4}\psi_{1,2}}{2\psi_{1,1}\psi_{3,2}} \end{aligned}$$

³If, for example, the coefficient $\psi_{3,2} = 2\sigma_X^2 = 1$, then the relative difference is 80 percent at $t = 0$ and then converging to zero as $t \rightarrow T$.

The solution $h_2(t)$ is then given by, for $\tau = \cdot\tau$:

$$\begin{aligned}
h_2(t) &= \frac{\Psi_{2,4}}{\psi_{2,1} + \psi_{3,1} + \psi_{1,1}} \left[\exp \left(-(\psi_{1,1} + \psi_{3,1})\tau \right) - \exp \left(-\psi_{2,1} \cdot \tau \right) \right] \\
&+ \frac{(\Psi_{2,3} - \Psi_{2,4})}{\psi_{3,1} + \psi_{2,1}} \left[\exp \left(-\psi_{3,1} \cdot \tau \right) - \exp \left(-\psi_{2,1} \cdot \tau \right) \right] \\
&+ \frac{\Psi_{2,4}}{\psi_{1,1} + \psi_{2,1}} \left[\exp \left(\psi_{2,1} \cdot \tau \right) - \exp \left(-\psi_{1,1} \cdot \tau \right) \right] \\
&+ \frac{\Psi_{2,4} - \Psi_{2,3} - \psi_{2,5}}{\psi_{2,1}} \left[1 - \exp \left(\psi_{2,1} \cdot \tau \right) \right].
\end{aligned} \tag{A.18}$$

Appendix B

Proofs and Complimentary Results for Chapter 2

B.1 Results and Proofs

In the derivations of the bond prices and bond price dynamics, two probability measures are used, P and Q . P denotes the physical measure, and Brownian motion under P are denoted as Z_i , where $i \in \{r, l\}$. The risk neutral probability measure is denoted as Q , and Brownian motions under Q are denoted as W_i , where $i \in \{r, l\}$. The formula for the bond price, given in Equation (3.7), shows that the dynamics of the interest rate and the liquidity under the measure Q is important. To get the Q -dynamics of r_t and l_t , I introduce the process

$$dW_t = dZ_t + \lambda_t dt \quad (\text{B.1})$$

for measurable functions $\lambda_t = [\lambda_r(t), \lambda_l(t)]$. The process dW is not a Brownian motion under the measure P , but from the Girsanov theorem, it is known that there exist a probability measure Q such that W_t is a Brownian motion with respect to Q . From Bjørk (2009), the probability measure Q is in fact a martingale measure for the bond market.

Bond Prices and Dynamics

Proposition B.1.1 *The price of a zero-coupon bond with maturity date T_1 is given by*

$$B(t, T_1, r_t, l_t) = \exp \left(A_1(t, T_1) - A_2(t, T_1)r_t - A_3(t, T_1)l_t \right) \quad (\text{B.2})$$

The function $A_1(t, T_1)$ is given by

$$\begin{aligned} A_1(t, T_1) = & \left[\left(\left(\frac{\sigma_r^2}{2\kappa_r^2} - \bar{r} \right) + \left(\frac{\sigma_l^2}{2\kappa_l^2} - \bar{l} \right) \right) (T_1 - t) + \left(\bar{r} - \frac{\sigma_r^2}{\kappa_r^2} \right) A_2(t, T_1) \right. \\ & + \left(\bar{l} - \frac{\sigma_l^2}{\kappa_l^2} \right) A_3(t, T_1) + \frac{\sigma_r^2}{4\kappa_r^3} (1 - e^{-2\kappa_r(T_1-t)}) \\ & \left. + \frac{\sigma_l^2}{4\kappa_l^3} (1 - e^{-2\kappa_l(T_1-t)}) + \int_t^{T_1} \sigma_B(t, T_1) \lambda ds \right] \end{aligned} \quad (\text{B.3})$$

whereas the functions $A_2(t, T_1)$ and $A_3(t, T_1)$ are given by

$$A_2(t, T_1) = \frac{1}{\kappa_r} (1 - e^{-\kappa_r(T_1-t)}) \quad (\text{B.4})$$

$$A_3(t, T_1) = \frac{1}{\kappa_l} (1 - e^{-\kappa_l(T_1-t)}). \quad (\text{B.5})$$

Proof: The risk-neutral description of the interest rate will be denoted as

$$dr_t = (\kappa_r(\bar{r} - r_t) + \sigma_r \lambda_r(t)) dt - \sigma_r dW_r, \quad (\text{B.6})$$

where κ_r describes the degree of mean reversion, \bar{r} is the long-run mean, σ_r is the interest rate volatility, and $\lambda_r(t)$ is the market price of interest rate risk. Similarly, the risk-neutral liquidity can be described by

$$dl_t = (\kappa_l(\bar{l} - l_t) + \sigma_l \lambda_l(t)) dt - \sigma_l dW_l. \quad (\text{B.7})$$

The process l_t is interpreted as the liquidity of the bond, due to how the bond price is affected by time-varying liquidity: an increase in liquidity should lead to a higher price as the investor must pay a premium for the option to resell the bond easier in the future. $\lambda_l(t)$ is the market price of liquidity risk. The interest rate and liquidity processes under Q , are found by using the Girsanov transformation given in Equation (B.1). Because, by the Girsanov theorem, $W_i(t)$ is a Brownian motion with respect to Q , I can replace Equation (3.7) with

$$B(t, T_1, r, l) = \mathbb{E}^t[\exp(-\int_t^{T_1} (\tilde{r}_s + \tilde{l}_s) ds)] \quad (\text{B.8})$$

where

$$\begin{aligned} d\tilde{r}(s) &= [\kappa_r(\bar{r} - \tilde{r}(s)) + \sigma_r \lambda_r(s)] ds - \sigma_r dW_r(s) \\ d\tilde{l}(s) &= [\kappa_l(\bar{l} - \tilde{l}(s)) + \sigma_l \lambda_l(s)] ds - \sigma_l dW_l(s). \end{aligned}$$

This gives , when $\tilde{r}(t) = r$ and $\tilde{l}(t) = l$,

$$\begin{aligned}\tilde{r}(s) &= e^{-\kappa_r(s-t)}r + \bar{r}(1 - e^{-\kappa_r(s-t)}) + \sigma_r \int_t^s e^{\kappa_r(u-s)}\lambda_r(u)du \\ &- \sigma_r \int_t^s e^{\kappa_r(u-s)}dW_r(u)\end{aligned}\tag{B.9}$$

$$\begin{aligned}\tilde{l}(s) &= e^{-\kappa_l(s-t)}l + \bar{l}(1 - e^{-\kappa_l(s-t)}) + \sigma_l \int_t^s e^{\kappa_l(u-s)}\lambda_l(u)du \\ &- \sigma_l \int_t^s e^{\kappa_l(u-s)}dW_l(u).\end{aligned}\tag{B.10}$$

If I now define $Y(t) = -\int_t^{T_1}(\tilde{r}_s + \tilde{l}_s)ds$, then $Y(t)$ is normally distributed with expectation equal to

$$\begin{aligned}\mathbb{E}[Y(t)] &= -\int_t^{T_1}(e^{-\kappa_r(s-t)}r + e^{-\kappa_l(s-t)}l)ds \\ &- \int_t^{T_1}(\bar{r}(1 - e^{-\kappa_r(s-t)}) + \bar{l}(1 - e^{-\kappa_l(s-t)}))ds \\ &+ \int_t^{T_1}\left(\frac{\sigma_r}{\kappa_r}(1 - e^{-\kappa_r(T_1-s)})\lambda_r(s)\right)ds \\ &+ \int_t^{T_1}\left(\frac{\sigma_l}{\kappa_l}(1 - e^{-\kappa_l(T_1-s)})\lambda_l(s)\right)ds\end{aligned}\tag{B.11}$$

and variance equal to

$$\mathbb{V}[Y(t)] = \int_t^{T_1}\left(\frac{\sigma_r^2}{\kappa_r^2}(e^{-\kappa_r(T_1-s)} - 1)^2 + \frac{\sigma_l^2}{\kappa_l^2}(e^{-\kappa_l(T_1-s)} - 1)^2\right)ds\tag{B.12}$$

I have applied the stochastic Fubini theorem and the Ito isometry. If a stochastic variable $Y_t \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[e^{Y(t)}] = e^{\mu + \frac{1}{2}\sigma^2}$. Thus, applying this result yields

$$\begin{aligned}B(t, T_1, r, l) &= \exp\left[\left(\left(\frac{\sigma_r^2}{2\kappa_r^2} - \bar{r}\right) + \left(\frac{\sigma_l^2}{2\kappa_l^2} - \bar{l}\right)\right)(T_1 - t) + \left(\bar{r} - \frac{\sigma_r^2}{2\kappa_r^2}\right)A_2(t, T_1) \right. \\ &+ \left(\bar{l} - \frac{\sigma_l^2}{2\kappa_l^2}\right)A_3(t, T_1) + \frac{\sigma_r^2}{4\kappa_r^3}(1 - e^{-2\kappa_r(T_1-t)}) \\ &+ \frac{\sigma_l^2}{4\kappa_l^3}(1 - e^{-2\kappa_l(T_1-t)}) + \int_t^{T_1}\sigma_r\lambda_r(s)A_2(s, T_1)ds \\ &+ \left.\int_t^{T_1}\sigma_l\lambda_l(s)A_3(s, T_1)ds - A_2(t, T_1)r - A_3(t, T_1)l\right] \\ &= \exp(A_1(t, T_1) - A_2(t, T_1)r_t - A_3(t, T_1)l_t)\end{aligned}\tag{B.13}$$

Which concludes the proof. \square

Proposition B.1.2 *If the interest rate and the liquidity are given by Equations (B.6) and (B.7), then the dynamic bond price under the measure Q is given by*

$$dB(t, T_1, r, l) = B(t, T_1) \left((r_t + l_t)dt - \sigma_r A_2(t, T_1) dW_r - \sigma_l A_3(t, T_1) dW_l \right) \quad (\text{B.14})$$

Proof:

The price of the bond is given by Equation (B.13), and can be rewritten as

$$B(t, T_1) = e^{Y(t, T_1)}$$

where $Y_t = A_1(t, T_1) - A_2(t, T_1)r_t - A_3(t, T_1)l_t$. In terms of integrals, the expression of $Y(t, T_1)$ is given by

$$\begin{aligned} Y(t, T_1) &= \int_t^{T_1} \left(\frac{\sigma_r^2}{2} A_2^2(s, T_1) + \frac{\sigma_l^2}{2} A_3^2(s, T_1) \right) ds \\ &\quad - \int_t^{T_1} \left(\kappa_r \bar{r} A_2(s, T_1) + \kappa_l \bar{l} A_3(s, T_1) \right) ds \\ &\quad + \int_t^{T_1} \left(\sigma_r \lambda_r(s) A_2(s, T_1) + \sigma_l \lambda_l(s) A_3(s, T_1) \right) ds \\ &\quad - A_2(t, T_1)r_t - A_3(t, T_1)l_t \end{aligned}$$

From Ito's lemma, the dynamical price of the bond is given by

$$dB(t, T_1, r, l) = B(t, T_1, r, l) dY_t + \frac{1}{2} B(t, T_1, r, l) (dY_t)^2 \quad (\text{B.15})$$

The expressions of $dY(t, T_1)$ and $(dY(t, T_1))^2$ must therefore be found. After standard rules of derivation under the integral sign, $dY(t, T_1)$ is given by

$$\begin{aligned} dY(t, T_1) &= - \left(\frac{\sigma_r^2}{2} A_2^2(t, T_1) + \frac{\sigma_l^2}{2} A_3^2(t, T_1) \right) dt + \left(\kappa_r \bar{r} A_2(t, T_1) + \kappa_l \bar{l} A_3(t, T_1) \right) dt \\ &\quad - \left(\sigma_r \lambda_r(t) A_2(t, T_1) + \sigma_l \lambda_l(t) A_3(t, T_1) \right) dt - A_2(t, T_1) dr - r dA_2(t, T_1) \\ &\quad - dr dA_2(t, T_1) - A_3(t, T_1) dl - l dA_3(t, T_1) - dl dA_3(t, T_1) \end{aligned}$$

The derivatives of the functions $A_i(t)$, with respect to t , are given by

$$\begin{aligned}
A'_1(t) &= -\left(\frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_l^2}{2\kappa_l^2} - \bar{r} - \bar{l}\right) - \left(\bar{r} - \frac{\sigma_r^2}{\kappa_r^2}\right)e^{-\kappa_r(T_1-t)} - \left(\bar{l} - \frac{\sigma_l^2}{\kappa_l^2}\right)e^{-\kappa_l(T_1-t)} \\
&\quad - \frac{\sigma_r^2}{2\kappa_r^2}e^{-2\kappa_r(T_1-t)} - \frac{\sigma_l^2}{2\kappa_l^2}e^{-2\kappa_l(T_1-t)} - \sigma_r\lambda_r(t)A_2(t) - \sigma_l\lambda_l(t)A_3(t) \\
A'_2(t) &= -e^{-\kappa_r(T_1-t)} \\
A'_3(t) &= -e^{-\kappa_l(T_1-t)}
\end{aligned}$$

Inserting the Q -dynamics of r and l and the expressions of $A'_2(t, T_1)$ and $A'_3(t, T_1)$ into the expression of dY , I get

$$\begin{aligned}
dY(t, T_1) &= -\left(\frac{\sigma_r^2}{2}A_2^2(t, T_1) + \frac{\sigma_l^2}{2}A_3^2(t, T_1)\right)dt + \left(\kappa_r\bar{r}A_2(t, T_1) + \kappa_l\bar{l}A_3(t, T_1)\right)dt \\
&\quad - \left(\sigma_r\lambda_r(t)A_2(t, T_1) + \sigma_l\lambda_l(t)A_3(t, T_1)\right)dt \\
&\quad - A_2(t, T_1)\left((\kappa_r(\bar{r} - r_t) - \sigma_r\lambda_r(t))dt + \sigma_r dW_r\right) + re^{-\kappa_r(T_1-t)}dt - 0 \\
&\quad - A_3(t, T_1)\left((\kappa_l(\bar{l} - l_t) - \sigma_l\lambda_l(t))dt + \sigma_l dW_l\right) + le^{-\kappa_l(T_1-t)}dt - 0
\end{aligned}$$

Since $\kappa_r A_2(t, T_1) = 1 - e^{-\kappa_r(T_1-t)}$ and $\kappa_l A_3(t, T_1) = 1 - e^{-\kappa_l(T_1-t)}$, I find the expression of dY_t is given by

$$dY_t = -\left(\frac{\sigma_r^2}{2}A_2^2(t, T_1) + \frac{\sigma_l^2}{2}A_3^2(t, T_1)\right)dt + r_t dt + l_t dt - \sigma_r A_2(t, T_1)dW_r - \sigma_l A_3(t, T_1)dW_l$$

To find $(dY_t)^2$, I apply the fact that

$$\begin{aligned}
dW_i dt &= (dZ_i + \lambda_i dt)dt = 0 \\
(dW_i)^2 &= dt
\end{aligned}$$

hence, since W_r and W_l are uncorrelated, it follows that¹

$$\begin{aligned}
(dY_t)^2 &= \left(-\left(\frac{\sigma_r^2}{2}A_2^2(t, T_1) + \frac{\sigma_l^2}{2}A_3^2(t, T_1)\right)dt + r_t dt + l_t dt \right. \\
&\quad \left. - \sigma_r A_2(t, T_1)dW_r - \sigma_l A_3(t, T_1)dW_l\right)^2.
\end{aligned}$$

¹The coefficients A_2 are functions of time, t , and the maturity of the bond, T_1 .

Which gives

$$(dY_t)^2 = \sigma_r^2 A_2^2(t, T_1)dt + \sigma_l^2 A_3^2(t, T_1)dt.$$

After substituting the expressions of dY and $(dY)^2$ into Equation (B.15), the risk-neutral dynamical equation for B_t is

$$\begin{aligned} dB(t, T_1, r, l) &= B(t, T_1) \left(- \left(\frac{\sigma_r^2}{2} A_2^2(t, T_1) + \frac{\sigma_l^2}{2} A_3^2(t, T_1) \right) dt + r_t dt \right. \\ &\quad + l_t dt - \sigma_r A_2(t, T_1) dW_r - \sigma_l A_3(t, T_1) dW_l \Big) \\ &\quad + \frac{1}{2} B(t, T_1) \left(\sigma_r^2 A_2^2(t, T_1) dt + \sigma_l^2 A_3^2(t, T_1) dt \right) \\ &= B(t, T_1) \left((r_t + l_t) dt - \sigma_r A_2(t, T_1) dW_r - \sigma_l A_3(t, T_1) dW_l \right) \end{aligned}$$

Which concludes the proof. \square

The next step is to derive the dynamical bond price under the measure P :

Proposition B.1.3 *If the bond price under the measure Q is given by Equation (B.14), then the dynamical bond price under P is given by*

$$dB_t = B(t, T_1) \left((r_t + l_t - \sigma_r \lambda_r A_2 - \sigma_l \lambda_l A_3) dt - \sigma_r A_1 dZ_r - \sigma_l A_2 dZ_l \right) \quad (\text{B.16})$$

Proof :

Using Proposition B.1.2 and the fact that $dW_t = dZ_t + \lambda_t dt$, I get

$$\begin{aligned} dB_t &= B(t, T_1) \left((r_t + l_t) dt - \sigma_r A_2(t, T_1) dW_r - \sigma_l A_3(t, T_1) dW_l \right) \\ &= B(t, T_1) \left((r_t + l_t) dt - \sigma_r A_2(t, T_1) (dZ_r + \lambda_r dt) - \sigma_l A_3(t, T_1) (dZ_l + \lambda_l dt) \right) \\ &= B(t, T_1) \left((r_t + l_t - \sigma_r \lambda_r A_2 - \sigma_l \lambda_l A_3) dt - \sigma_r A_1 dZ_r - \sigma_l A_2 dZ_l \right) \end{aligned}$$

Which concludes the proof. \square

I now show how the wealth dynamics are derived.

Proposition B.1.4 *If the bond price is given by Equation (B.16), then the dynamical wealth equation under P is given by*

$$dw = \left(r + \pi_t (l - \sigma_r \lambda_r(t) A_2(t) - \sigma_l \lambda_l(t) A_3(t)) \right) w_t dt - \pi_t w_t (\sigma_r A_2(t) dZ_1 + \sigma_l A_3(t) dZ_2) \quad (\text{B.17})$$

Proof: The wealth is given by the sum of what is invested in the risk-less asset, x , and the risky asset, y . The dynamical expressions of those amounts are given by dx and dy , thus

from the fact that the bank account evolves according to

$$dR = r_t R dt \quad (\text{B.18})$$

and Equation (B.16)

$$\begin{aligned} dw &= dx + dy \\ &= r_x dt + y_t [(r_t + l_t - \sigma_r \lambda_r(t) A_2(t, T_1) - \sigma_l \lambda_l(t) A_3(t, T_1)) dt \\ &\quad - (\sigma_r A_2(t, T_1) dZ_1 + \sigma_l A_3(t, T_1) dZ_2)] \\ &= r \frac{x}{w} w dt + \frac{y_t}{w} w [(r_t + l_t - \sigma_r \lambda_r(t) A_2(t, T_1) - \sigma_l \lambda_l(t) A_3(t, T_1)) dt \\ &\quad - (\sigma_r A_2(t, T_1) dZ_1 + \sigma_l A_3(t, T_1) dZ_2)] \\ &= r(1 - \pi_t) w dt + \pi_t w [(r_t + l_t - \sigma_r \lambda_r(t) A_2(t, T_1) - \sigma_l \lambda_l(t) A_3(t, T_1)) dt \\ &\quad - (\sigma_r A_2(t, T_1) dZ_1 + \sigma_l A_3(t, T_1) dZ_2)] \\ &= \left(r + \pi_t (l - \sigma_r \lambda_r(t) A_2(t) - \sigma_l \lambda_l(t) A_3(t)) \right) w_t dt \\ &\quad - \pi_t w_t (\sigma_r A_2(t) dZ_1 + \sigma_l A_3(t) dZ_2) \end{aligned}$$

Which concludes the proof. \square

Proof of Proposition (3.4.1):

From the fact that $dw = dx + dy$, where dx and dy are the evolution of the amounts invested in the bank and the risky bond, the HJB equation is given by

$$\begin{aligned} 0 &= \phi_t + (r + \pi(l + M_1)) w \phi_w + \kappa_r (\bar{r} - r) \phi_r + \kappa_l (\bar{l} - l) \phi_l \\ &\quad + \frac{w^2}{2} \pi^2 M_2 \phi_{ww} + \frac{\sigma_r^2}{2} \phi_{rr} + \frac{\sigma_l^2}{2} \phi_{ll} - \pi w M_3 \phi_{rw} - \pi w M_4 \phi_{lw} \end{aligned}$$

The first order condition with respect to π is

$$\begin{aligned} \pi^* &= \frac{\sigma_r^2 A_2(t) \phi_{rw} + \sigma_l^2 A_3(t) \phi_{lw} - (l - \sigma_r \lambda_r A_2(t) - \sigma_l \lambda_l A_3(t)) \phi_w}{w(\sigma_r^2 A_2(t)^2 + \sigma_l^2 A_3(t)^2) \phi_{ww}} \\ &= \frac{M_3}{M_2} \frac{\phi_{rw}}{w \phi_{ww}} + \frac{M_4}{M_2} \frac{\phi_{lw}}{w \phi_{ww}} - \frac{l + M_1}{M_2} \frac{\phi_w}{w \phi_{ww}} \\ &= \pi_r + \pi_l + \pi_m \end{aligned}$$

Assume now that the value function ϕ_t can be separated into the product

$$\phi(t, w, r, l) = \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1}(t, r) h^{\theta_2}(t, l)$$

so the partial derivatives of the value function can be written as

$$\begin{aligned} \phi_t &= \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \left[\theta_1 \frac{f_t}{f} + \theta_2 \frac{h_t}{h} \right] \\ \phi_r &= \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \theta_1 \frac{f_r}{f}, \quad \phi_{rr} = \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \left[\theta_1 (\theta_1 - 1) \frac{f_r^2}{f^2} + \theta_1 \frac{f_{rr}}{f} \right] \\ \phi_l &= \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \frac{h_l}{h} \theta_2, \quad \phi_{ll} = \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \left[\theta_2 (\theta_2 - 1) \frac{h_l^2}{h^2} + \theta_2 \frac{h_{ll}}{h} \right] \\ \phi_w &= \frac{1-\gamma}{w} \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2}, \quad \phi_{ww} = -\gamma \frac{1-\gamma}{w^2} \frac{w^{1-\gamma}}{1-\gamma} f^{\theta_1} h^{\theta_2} \\ \phi_{wr} &= w^{-\gamma} f^{\theta_1} h^{\theta_2} \theta_1 \frac{f_r}{f}, \quad \phi_{wl} = w^{-\gamma} f^{\theta_1} h^{\theta_2} \theta_2 \frac{h_l}{h}. \end{aligned}$$

The HJB equation then, after multiplying by $\frac{1-\gamma}{w^{1-\gamma} f^{\theta_1} h^{\theta_2}}$, becomes

$$\begin{aligned} 0 &= \theta_1 \frac{f_t}{f} + \theta_2 \frac{h_t}{h} + r(1-\gamma) + \kappa_l(\bar{l} - l) \theta_2 \frac{h_l}{h} + \kappa_r(\bar{r} - r) \theta_1 \frac{f_r}{f} \\ &+ \frac{1-\gamma}{2\gamma} \frac{(l + M_1)^2}{M_2} - \theta_1(l + M_1) \frac{1-\gamma}{\gamma} \frac{M_3}{M_2} \frac{f_r}{f} - \theta_2(l + M_1) \frac{1-\gamma}{\gamma} \frac{M_4}{M_2} \frac{h_l}{h} \\ &+ \theta_2(\theta_2 - 1) \frac{\sigma_l^2}{2} \frac{h_l^2}{h^2} + \theta_2 \frac{\sigma_l^2}{2} \frac{h_{ll}}{h} + \theta_1(\theta_1 - 1) \frac{\sigma_r^2}{2} \frac{f_r^2}{f^2} + \theta_1 \frac{\sigma_r^2}{2} \frac{f_{rr}}{f} \\ &+ \theta_1^2 \frac{1-\gamma}{2\gamma} \frac{M_3^2}{M_2} \frac{f_r^2}{f^2} + \theta_2^2 \frac{1-\gamma}{2\gamma} \frac{M_4^2}{M_2} \frac{h_l^2}{h^2} \\ &+ \theta_1 \theta_2 \frac{1-\gamma}{\gamma} \frac{M_3 M_4}{M_2} \frac{f_r h_l}{f h} \end{aligned}$$

To remove some of the most severe nonlinearities in this differential equation, remember Assumption 1:

$$K_1 = \frac{M_3^2}{M_2}, \quad K_2 = \frac{M_4^2}{M_2}$$

where the coefficients θ_1 and θ_2 equal

$$\theta_1 = \frac{\sigma^2}{\sigma^2 + \frac{1-\gamma}{\gamma} K_1}, \quad \theta_2 = \frac{\eta^2}{\eta^2 + \frac{1-\gamma}{\gamma} K_2}.$$

Under this assumption, the HJB equation becomes

$$\begin{aligned}
0 &= \theta_1 \frac{f_t}{f} + \theta_2 \frac{h_t}{h} + r(1 - \gamma) + \kappa_l(\bar{l} - l)\theta_2 \frac{h_l}{h} + \kappa_r(\bar{r} - r)\theta_1 \frac{f_r}{f} \\
&+ \frac{1 - \gamma}{2\gamma} \frac{(l + M_1)^2}{M_2} - \theta_1(l + M_1) \frac{1 - \gamma}{\gamma} \frac{M_3}{M_2} \frac{f_r}{f} - \theta_2(l + M_1) \frac{1 - \gamma}{\gamma} \frac{M_4}{M_2} \frac{h_l}{h} \\
&+ \theta_2 \frac{\sigma_l^2}{2} \frac{h_{ll}}{h} + \theta_1 \frac{\sigma_r^2}{2} \frac{f_{rr}}{f} + \theta_1 \theta_2 \frac{1 - \gamma}{\gamma} \frac{M_3 M_4}{M_2} \frac{f_r h_l}{f h}.
\end{aligned}$$

Thus, if the functions f and h are given by

$$f(t, r) = \exp(g_1(t) + g_2(t)r), \quad h(t, l) = \exp(g_3(t)l + g_4(t)l^2)$$

then the partial derivatives are

$$\begin{aligned}
f_t &= (g'_1(t) + g'_2(t)r)f, & f_r &= g_2(t)f, & f_{rr} &= g_2^2(t)f \\
h_t &= (g'_3(t)l + g'_4(t)l^2)h, & h_l &= (g_3(t) + 2g_4(t)l)h \\
h_{ll} &= \left[g_3^2(t)2g_4(t) + 4g_3(t)g_4(t)l + 4g_4(t)l^2 \right] h.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{f_t}{f} &= g'_1(t) + g'_2(t)r, & \frac{h_t}{h} &= g'_3(t)l + g'_4(t)l^2, & \frac{f_r}{f} &= g_2(t) & \frac{f_{rr}}{f} &= g_2^2(t) \\
\frac{h_l}{h} &= g_3(t) + 2g_4(t)l, & \frac{h_{ll}}{h} &= g_3^2(t)2g_4(t) + 4g_3(t)g_4(t)l + 4g_4(t)l^2 \\
\frac{f_r h_l}{f h} &= g_2(t) \left[g_3(t) + 2g_4(t)l \right].
\end{aligned}$$

If I now substitute these expressions into the HJB equation and collect all terms with

similar expression for either r or l , I get

$$\begin{aligned}
0 = & \left[\theta_1 g'_1(t) + \kappa_l \bar{l} \theta_2 g_3 + \kappa_r \bar{r} \theta_1 g_2 + \frac{1-\gamma}{2\gamma} \frac{M_1^2}{M_2} - \theta_1 M_1 \frac{1-\gamma}{\gamma} \frac{M_3}{M_2} g_2 \right. \\
& - \left. \theta_2 M_1 \frac{1-\gamma}{\gamma} \frac{M_4}{M_2} g_3 + \theta_1 \frac{\sigma_r^2}{2} g_2^2 + \theta_2 \sigma_l^2 g_3^2 g_4 + \theta_1 \theta_2 \frac{1-\gamma}{\gamma} \frac{M_3 M_4}{M_2} g_2 g_3 \right] \\
& + r \left[\theta_1 g'_2 - (1-\gamma) + \kappa_r \theta_1 g_2 \right] \\
& + l \left[\theta_2 g'_3 - \kappa_l \theta_2 g_3 + \frac{1-\gamma}{\gamma} \frac{M_1}{M_2} - \theta_1 \frac{1-\gamma}{\gamma} \frac{M_3}{M_2} g_2 - \theta_2 \frac{1-\gamma}{\gamma} \frac{M_4}{M_2} g_3 \right. \\
& + \left. 2\theta_2 \sigma_l^2 g_3 g_4 + 2\theta_1 \theta_2 \frac{1-\gamma}{\gamma} \frac{M_3 M_4}{M_2} g_2 g_4 \right] \\
& + l^2 \left[\theta_2 g'_4 - 2\theta_2 \left[\kappa_l + \frac{1-\gamma}{\gamma} \frac{M_4}{M_2} - \sigma_l^2 \right] g_4(t) + \frac{1-\gamma}{2\gamma} \frac{1}{M_2} \right]. \tag{B.19}
\end{aligned}$$

For the HJB to equal zero, all terms given in the squared brackets must equal zero. The solutions to the differential equations can be solved and the proof is complete. \square

B.1.1 Approximations of the Functions $g_i(t)$

To analyze the investment strategies, I need to solve the differential equations $g_i(t)$. In this respect, I will make some simple approximations in order to find how much the investor should invest in the bond during the lifetime of the investment period. The solution of the differential equation $g'_2(t)$ given in line three of Equation (B.19) is straightforward, because it is given by an ordinary differential equation. Specifically, it is given by

$$g_2(t) = \frac{1-\gamma}{\kappa_r \theta_1} \left[1 - \exp(-\kappa_r(T-t)) \right] \tag{B.20}$$

To find a solution to the function $g'_4(t)$, it is necessary to approximate the ratio $\frac{M_4}{M_2}$. This is done with a Taylor expansion, and in particular I get, by expanding around $t = a$:

$$\begin{aligned}
\frac{M_4}{M_2} &= \frac{\sigma_l^2 A_3(a)}{\sigma_l^2 A_3^2(a) + \sigma^2 A_2^2(a)} - \frac{\sigma_l^2 A'_3(a) - 2 \frac{\sigma_l^2 \kappa_l^2 \kappa_r^2 A_3(a)(\sigma_l^2 A_3(a) A'_3(a) + \sigma_r^2 A_2(a) A'_2(a))}{\sigma_l^2 \kappa_r^2 A_3^2(a) + \sigma_r^2 \kappa_l^2 A_2^2(a)}}{\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a)} (t-a) \\
&= k_1 + k_2(t-a)
\end{aligned}$$

A Taylor expansion of $\frac{1}{M_2}$ is given by

$$\begin{aligned}\frac{1}{M_2} &= \frac{1}{\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a)} + \frac{2\kappa_l^2 \kappa_r^2 (\sigma_l^2 A_3(a) A_3'(a) + \sigma_r^2 A_2(a) A_2'(a)) (t-a)}{(\sigma_l^2 \kappa_r^2 A_3^2(a) + \sigma_r^2 \kappa_l^2 A_2^2(a)) (\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a))} \\ &= k_3 + k_4(t-a)\end{aligned}$$

Applying these approximations, the differential equation $g_4'(t)$ has the solution

$$g_4(t) = \frac{\beta_4}{\beta_2} \left(1 - e^{\frac{T-t}{2}(\beta_2(T-t)+2\beta_1)} \right)$$

Where the coefficients β_1 , β_2 , β_3 , and β_4 are given by

$$\begin{aligned}\beta_1 &= \kappa_l - \sigma_l^2 + \frac{1-\gamma}{\gamma}(k_1 - k_2 a), \quad \beta_2 = \frac{1-\gamma}{\gamma} k_2 \\ \beta_3 &= -\frac{1-\gamma}{2\theta_2 \gamma}(k_3 + k_4 a), \quad \beta_4 = \frac{1-\gamma}{2\theta_2 \gamma} k_4\end{aligned}$$

where a is the constant for which the Taylor expansion is made around.

To find a solution to the function $g_3'(t)$, I approximate the ratio $\frac{M_3 M_4}{M_2}$, again via Taylor expansions around $t = a$, applying $\tau_a = T - a$:

$$\begin{aligned}\frac{M_3 M_4}{M_2} &= \left[\frac{A_3(a) A_2(a)}{\sigma_l^2 A_3(a) + \sigma_r^2 A_2(a)} - \frac{(t-a)}{\sigma_l^2 A_3(a) + \sigma_r^2 A_2(a)} \left[\frac{A_3(a)}{e^{\kappa_r \tau_a}} + \frac{A_2(a)}{e^{\kappa_l \tau_a}} \right. \right. \\ &\quad \left. \left. - 2\kappa_r \kappa_l \frac{A_2(a) A_3(a) (\sigma_l^2 A_3(a) e^{-\kappa_l \tau_a} + \sigma_r^2 A_2(a) e^{-\kappa_r \tau_a})}{\sigma_l^2 \kappa_r^2 A_3(a)^2 + \sigma_r^2 \kappa_l^2 A_2(a)^2} \right] \right] \sigma_l^2 \sigma_r^2 \\ &= k_5 + k_6(t-a)\end{aligned}$$

and the ratio $\frac{M_3}{M_2}$ is given by

$$\begin{aligned}\frac{M_3}{M_2} &= \frac{\sigma_r^2 A_2(a)}{\sigma_r^2 A_2^2(a) + \sigma_l^2 A_3^2(a)} \\ &\quad - \frac{\sigma_r^2 A_2'(a) + 2 \frac{\sigma_r^2 \kappa_r^2 \kappa_l^2 A_2(a) (\sigma_l^2 A_3(a) A_3'(a) + \sigma_r^2 A_2(a) A_2'(a))}{\sigma_l^2 \kappa_r^2 A_3^2(a) + \sigma_r^2 \kappa_l^2 A_2^2(a)}}{\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a)} (t-a). \\ &= k_7 + k_8(t-a)\end{aligned}$$

The ratio, $\frac{M_1}{M_2}$ is approximated by

$$\begin{aligned}
\frac{M_1}{M_2} &= -\frac{\sigma_r \lambda_r A_2(a) + \sigma_l \lambda_l A_3(a)}{\sigma_r^2 A_2^2(a) + \sigma_l^2 A_3^2(a)} \\
&+ \frac{\left(\sigma_r \lambda_r A_2'(a) + \sigma_l \lambda_l A_3'(a)\right)(t-a)}{\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a)} \\
&- \frac{2 \frac{\kappa_l^2 \kappa_r^2 (\sigma_r \lambda_r A_2(a) + \sigma_l \lambda_l A_3(a)) (\sigma_l^2 A_3(a) A_3'(a) + \sigma_r^2 A_2(a) A_2'(a))}{\sigma_r^2 \kappa_r^2 A_3^2(a) + \sigma_l^2 \kappa_l^2 A_2^2(a)}}{\sigma_l^2 A_3^2(a) + \sigma_r^2 A_2^2(a)} (t-a) \\
&= k_9 + k_{10}(t-a)
\end{aligned}$$

Gathering the preceding approximations, the differential equation $g_3'(t)$ can be written as

$$0 = g_3'(t) + (\beta_5 + \beta_6 t)g_3(t) + (\beta_7 + \beta_8 t)g_2(t) - \beta_9 - \beta_{10}t$$

where the coefficients β_i are given by

$$\begin{aligned}
\beta_5 &= \kappa_l + \frac{1-\gamma}{\gamma}(k_1 - k_2 a), & \beta_6 &= \frac{1-\gamma}{\gamma}k_2 \\
\beta_7 &= -\frac{\theta_1}{\theta_2} \frac{1-\gamma}{\gamma}(k_7 - k_8 a), & \beta_8 &= -\frac{\theta_1}{\theta_2} \frac{1-\gamma}{\gamma}k_8 \\
\beta_9 &= \frac{1-\gamma}{\theta_2 \gamma}(k_9 - k_{10} a), & \beta_{10} &= \frac{1-\gamma}{\theta_2 \gamma}k_{10}
\end{aligned}$$

When applying the estimates of the parameters governing the bond price, the interest rate and the market liquidity, given in table I, to the coefficients k_i 's and β_i 's, I find that the magnitude of some of these coefficients are negligible compared to the others. The approximated solution to the differential equation $g_3'(t)$ is therefore

$$\begin{aligned}
g_3(t) &= \left[\frac{\beta_7(1-\gamma)}{\kappa_r + \beta_5}(1 - e^{(\kappa_r + \beta_5)(T-t)}) \right. \\
&+ \left. \frac{\beta_9 \kappa_r - (1-\gamma)\beta_7}{\beta_5}(1 - e^{\beta_5(T-t)}) \right] \frac{e^{-\beta_5(T-t)}}{\kappa_r \theta_1}
\end{aligned}$$

If I now substitute all these expressions into the optimal trading policy, I can analyze how much should be invested in the bond at all times during the investment period.

Chapter 4

Conclusion and Future Work

4.1 Conclusion of the Thesis

Despite the tremendous amount of work on portfolio allocation, stock return predictability, and market liquidity, there are few papers investigating the impact of stochastic interest rates combined with either predictability, or time-varying market liquidity, on optimal asset allocation. In the paper *Life-Cycle Asset Allocation Under Stochastic Interest Rates and Stock Return Predictability*, stochastic real rates and a time-varying risk premium for stocks are tied together. An analytical solution to the problem is provided, and it proves that the proposed value function and corresponding portfolio are indeed the optimal solutions to the asset allocation problem. In another paper, *Optimal Portfolio Choice under Time-Varying Interest Rates and Bid-Ask Spreads*, the impact of stochastic market liquidity on the optimal investments in bonds is investigated. The following summarizes the key findings of the two chapters of the thesis.

The first chapter, *Life-Cycle Asset Allocation Under Stochastic Interest Rates and Stock Return Predictability*, analyzes the optimal asset allocation of a finite horizon investor who has three asset classes to invest in: cash, bonds, and stocks. It is assumed that the interest rate and risk premium for stocks are stochastic. This has significant impact on the optimal asset allocation. First, the optimal portfolio is derived in closed form. This is carried out via a verification theorem. Second, the optimal fractions allocated to bonds and stocks are analyzed. Because the investor can predict stock returns, the fraction of wealth allocated to stocks might increase in time. This contrasts previous results on the matter. However, from the perspective of the investor, this makes sense: if there is a variable that affects an asset's return, such that it is expected to increase in the future, one should increase investments in this asset. This implies that traditional portfolio advice,

which suggests decreasing exposure to stocks over time, is not necessarily valid for an investor who can predict returns.

There are several possible ways to extend the research in this chapter. The most promising extension from a general finance perspective, is to include a time-varying bond-stock correlation. This requires specifying a time-varying correlation process, and the Wishart process would be an obvious choice.

In the second chapter, *Optimal Portfolio Choice under Time-Varying Interest Rates and Bid-Ask Spreads*, the impact of time-varying market liquidity on the optimal asset allocation is analyzed. In the empirical part of the paper, it is found that the high-low spread estimator is mean reverting with considerable volatility. Further, the indirect utility function and the corresponding optimal trading policy are in closed form. The trading strategy is divided into three components: the mean-variance component, an interest rate hedging component, and a liquidity risk hedging component. All these components are time-dependent. The two hedging terms converge to zero in time, and they equal zero at the investment horizon. Moreover, it is found that the spot level of the bid-ask spread is included in the optimal trading strategy.

A drawback of the particular liquidity measure applied in this chapter is that it produces a substantial amount of negative estimates, an undesirable feature of the bid-ask spread. The problem with negative numbers is present in all existing estimators of the bid-ask spread: the Roll measure of Roll (1984), Lesmond et al. (1999), and the Gibbs estimator of Hasbrouck (2009). A venue for future research could be developing a non-negative bid-ask spread estimator, as it would be of interest not only for asset allocation problems, but also more broadly for general finance problems.

Index

- Z_l , 84
- Z_r , 84
- α , 80
- \bar{X} , 31
- \bar{r} , 25, 84
- β , 29, 79
- γ , 33
- κ , 25
- κ_X , 31
- κ_r , 84
- π_B , 36
- π_S , 36
- ρ_{SB} , 52
- σ_S , 29
- σ_X , 31
- σ_r , 25, 84
- θ_1 , 35
- θ_2 , 35
- ξ , 79
- τ , 26
- Asset allocation puzzle, 54
- Assumption 3.4.1, 89
- Bid-ask spread, 77
- Bond, 25
- Bond price under P , 29
- Bond price under Q , 27
- Bond-to-stock ratio, 54
- BSR, 54
- Cole-Hopf Transformation, 35
- Correlation between stocks and bonds, 52
- Credit rating, 84
- Dividend-Price ratio, 30
- Dow Jones Industrial Average, 74
- DPR, 30, 37, 44
- Dynamic asset allocation, 37
- FINRA, 73
- flight-to-quality, 52
- Girsanov's Theorem, 27
- Hamilton-Jacobi-Bellmann, 34, 88
- Indirect utility, 88
- Indirect value function, 35
- Interest rate, 25
- Investor Preferences, 33
- Investor preferences, 87
- Labor-Income to Consumption ratio, 30
- Lemma 2.2.1, 29
- LICR, 30, 41
- Liquidity, 80
- Liquidity augmented bond price, 85
- Liquidity process, 82
- Market liquidity, 80
- Market price of risk, 26
- Market price of risk, vector, 30

Optimal portfolios, 47

Predictor, 30

Price cushion, 77

Price dynamics, 30

Proposition 2.2.1, Chapter 1, 26

Proposition 2.2.2, 31

Proposition 3.4.1, 89

Robert Shiller, 39

Self-financing portfolio, 31

Time horizon, 48

Time to maturity, 26

Time-series of liquidity, 81

Time-varying liquidity, 68

TRACE, 73

Undervalued, 53

Vasicek, 25

Wealth, 31

Wealth dynamics, 31

Zero coupon bond, 26

Bibliography

- Aase, K. (1984). Optimum portfolio diversification in a general continuous-time model. *Stochastic Processes and their Applications* 18, 81–98.
- Acharya, V. and L. Pedersen (2005). Asset pricing and liquidity risk. *Journal of Financial Economics* 77, 375–410.
- Amihud, Y., V. Acharya, and S. Bharath. (2013). Liquidity risk of corporate bond returns: A conditional approach. *Journal of Financial Economics Forthcoming*.
- Amihud, Y. and H. Mendelson (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics* 17(2), 223–249.
- Amihud, Y. and H. Mendelson (1989). The effects of beta, bid-ask spread, residual risk and size on stock returns. *The Journal of Finance* 44, 479–486.
- Amihud, Y. and H. Mendelson (1991). Liquidity, maturity, and the yields on u.s. treasury securities. *The Journal of Finance* 46, 1411–1425.
- Ang, A. and G. Bekaert (2007, May). Stock return predictability: Is it there? *The Review of Financial Studies* 20(3), 651–707.
- Avramov, D. (2004). Stock return predictability and asset pricing models. *The Review of Financial Studies* 17(3), 699–738.
- Bajeux-Besnaiou, I., J. Jordan, and R. Portait (2001). An asset allocation puzzle: Comment. *The American Economic Review* 91, 1170–1179.
- Bao, J., J. Pan, and J. Wang (2011). The illiquidity of corporate bonds. *The Journal of Finance* 3(66), 911–946.
- Barberis, N. (2000). Investing for the long run when returns are predictable. *The Journal of Finance* 55, 225–264.
- Barberis, N., A. Shleifer, and J. Wurgler (2005). Comovement. *Journal of Financial Economics* 75(2), 283–317.

- Bartle, R., G. (1995). *The Elements of Integration and Lebesgue Measure*. Wiley.
- Benth, F. and K. Karlsen (2005). A note on merton's portfolio selection problem for the schwartz meanreversion model. *Stochastic Analysis and Applications* 23(4), 687–704.
- Benth, F., K. Karlsen, and K. Reikvam (2001). Optimal portfolio management rules in a non-gaussian market with durability and intertemporal substitution. *Finance and Stochastics* 5(4).
- Benzoni, L., P. Collin-Dufresne, and R. Goldstein (2007, Oct). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance* 62(5), 2123–2167.
- Bessembinder, H. and K. Venkataraman (2009). Bid-ask spreads: Measuring trade execution costs in financial markets. *Encyclopedia of quantitative finance*, 184–190.
- Björk, T. (2009). *Arbitrage theory in continuous time*. Oxford University Press.
- Brennan, M. J., E. S. Schwartz, and R. Lagnado (1997). Strategic asset allocation. *Journal of Economic Dynamics and Control* 21, 1377–1403.
- Brennan, M. J. and Y. Xia (2002). Dynamic asset allocation under inflation. *The Journal of Finance* LVII(3), 1201–1238.
- Brunnermeier, M. and L. Pedersen (2009). Market liquidity and funding liquidity. *The Review of Financial Studies* 22(6), 2201–2238.
- Buraschi, A., P. Porchia, and F. Trojani (2010). Correlation risk and optimal portfolio choice. *The Journal of Finance* LXV(1), 393–420.
- Campbell, J. Y. and J. Ammer (1993). What moves the stock and bond markets? a variance decomposition for long-term asset returns. *The Journal of Finance* 48(1), 3–37.
- Campbell, J. Y. and R. J. Shiller (1988). The dividend-price ratio and expectations of future dividends and discount factors. *The Review of Financial Studies* 1(3), 195–228.
- Campbell, J. Y. and S. B. Thompson (2008). Predicting excess stock returns out of sample: Can anything beat the historical average? *The Review of Financial Studies* 21(4), 1509–1531.
- Campbell, J. Y. and L. Viceira (1999). Consumption and portfolio decisions when expected returns are time varying. *The Quarterly Journal of Economics* 114(2), 433–495.

- Canner, N., N. Mankiw, and D. Weil (1997). An asset allocation puzzle. *The American Economic Review* 87, 181–191.
- Chacko, G. and L. Viceira (2005). Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *The Review of Financial Studies* 18(4), 1369–1402.
- Chen, L., D. A. Lesmond, and J. Wei (2007). Corporate yield spreads and bond liquidity. *The Journal of Finance* 62(1), 119–149.
- Chordia, T., R. Roll, and A. Subrahmanyam (2000). Commonality in liquidity. *Journal of Financial Economics* 59(1), 3–32.
- Christiansen, C. and A. Rinaldo (2007). Realized bond—stock correlation: Macroeconomic announcement effects. *Journal of Futures Markets* 27(5), 439–469.
- Cochrane, J. H. (2008). The dog that did not bark: a defense of return predictability. *The Review of Financial Studies* 21(4), 1533–1575.
- Connolly, R., C. Stivers, and L. Sun (2005). Stock market uncertainty and the stock-bond return relation. *Journal of Financial and Quantitative Analysis* 40, 161–194.
- Conroy, R. M., R. S. Harris, and B. A. Benet (1990). The effects of stock splits on bid-ask spreads. *The Journal of Finance* 45(4), 1285–1295.
- Corwin, S. and P. Schultz (2012, April). A simple way to estimate bid-ask spreads from daily high and low prices. *The Journal of Finance* 67(2), 719–759.
- Dai, Q. and K. Singleton (2000). Specification analysis of affine term structure models. *The Journal of Finance* 55(5), 1943–1978.
- Duffie, D. and R. Kan (1996). Modeling term structures of interest rates. *Mathematical Finance* 6, 379–406.
- Duffie, D. and K. Singleton (1999). Modeling term structures of defaultable bonds. *The Review of Financial Studies* 12, 687–720.
- Edwards, A., H. Lawrence, and M. Piwowar (2007). Corporate bond market transaction costs and transparency. *The Journal of Finance* 62(3), 1421–1451.
- Fontaine, J.-S. and R. Garcia (2012). Bond liquidity premia. *The Review of Financial Studies* 25(4), 1207–1254.
- Foucault, T., O. Kadan, and E. Kandel (2013). Liquidity cycles and make/take fees in electronic markets. *The Journal of Finance* 68(1), 299–341.
- Grinold, R. and R. Kahn (1999). *Active Portfolio Management*. McGraw Hill.

- Hasbrouck, J. (2009). Trading costs and returns for u.s. equities: Estimating effective costs from daily data. *The Journal of Finance* 64(3), 1445–1477.
- Hasbrouck, J. and D. Seppi (2001). Common factors in prices, order flows and liquidity. *Journal of Financial Economics* 59, 383–411.
- Honda, T. and S. Kamimura (2011). On the verification theorem of continuous-time optimal portfolio problems with stochastic market price of risk. *Asia-pacific financial markets* 18, 151–166.
- Huang, R. and H. Stoll (1997). The components of the bid-ask spread: a general approach. *The Review of Financial Studies* 10(4), 995–1034.
- Ilmanen, A. (2003). Stock-bond correlations. *The Journal of Fixed Income* 13(2), 55–66.
- Ingersoll, J. E. J. (1987). *Theory of Financial Decision Making*. Rowman & Littlefield.
- Kim, T. S. and E. Omberg (1996). Nonmyopic portfolio behavior. *The Review of Financial Studies* 9(1), 141–161.
- Korn, R. and H. Kraft (2001). A stochastic control approach to portfolio problems with stochastic interest rates. *SIAM Journal of Control and Optimization* 40, 1250–1269.
- Korn, R. and H. Kraft (2004). On the stability of continuous-time portfolio problems with stochastic opportunity set. *Mathematical Finance* 14, 403–413.
- Kraft, H. (2003). *Optimal Portfolios with Stochastic Interest Rates and Defaultable Assets*. Lecture notes in economics and mathematical systems. Springer.
- Kraft, H., F. Seifried, and M. Steffensen (2013). Consumption-portfolio optimization with recursive utility in incomplete markets. *Finance and Stochastics* 17(1), 161–196.
- Langetieg, T. (1980). A multivariate model of the term structure. *The Journal of Finance* 1, 71–97.
- Lesmond, D., J. Ogden, and C. Trzcinka (1999). A new estimate of transaction costs. *The Review of Financial Studies* 12(5), 1113–1141.
- Liu, J. (2007). Portfolio selection in stochastic environments. *The Review of Financial Studies* 20(1), 1–39.
- Liu, J., F. Longstaff, and J. Pan (2003). Dynamic asset allocation with event risk. *The Journal of Finance* LVIII(1), 231–259.
- Menzly, L. and P. Veronesi (2004). Understanding predictability. *The Journal of Political Economy* 112(1), 1–47.

- Merton, R. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics* 51, 247–257.
- Merton, R. (1971). Optimal consumption and portfolio rules in a continuous time model. *Journal of Economic Theory* 3, 373–413.
- Munk, C. and C. Sørensen (2010). Dynamic asset allocation with stochastic income and interest rates. *Journal of Financial Economics* 96, 433–462.
- Næs, R., J. Skjeltorp, and B. Ødegaard (2011, February). Stock market liquidity and the business cycle. *The Journal of Finance* 66(1), 139–176.
- Øksendal, B. (2003). *Stochastic Differential Equations, An introduction with applications*. Springer.
- Pedersen, L. (2009). When everyone runs for the exit. *International journal of central banking* 5, 177–199.
- Perold, A. F. (1988). The implementation shortfall: paper vs. reality. *The Journal of Portfolio Management* 14, 4–9.
- Polyanin, A. D. and V. F. Zaitsev (2003). *Handbook of Exact Solutions for Ordinary Differential Equations* (2nd ed.). Chapman & Hall/CRC, Boca Raton.
- Puhle, M. (2007). *Bond Portfolio Optimization*. Lecture notes in economics and mathematical systems. Springer.
- Ringer, N. and M. Tehranchi (2006). Optimal portfolio choice in the bond market. *Finance and Stochastics* 10(4).
- Roll, R. (1984). A simple implicit measure of the effective bid-ask spread in an efficient market. *The Journal of Finance* 39(4), 1127–1140.
- Sangvinatsos, A. and J. Wachter (2005). Does the failure of the expectations hypothesis matter for long-term investors? *The Journal of Finance* LX(1), 179–230.
- Santos, T. and P. Veronesi (2006). Labor income and predictable stock returns. *The Review of Financial Studies* 19(1), 1–44.
- Shiller, R. J. and A. Baltratti (1992). Stock prices and bond yields: Can their com-movements be explained in terms of present value models? *The Journal of Monetary Economics* 30, 25–46.
- Sørensen, C. (1999). Dynamic asset allocation under and fixed income management. *Journal of Financial and Quantitative Analysis* 34, 513–531.

- Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* 5, 177–188.
- Viceira, L. (2001, April). Optimal portfolio choice for longhorizon investors with non-tradable labor income. *The Journal of Finance* 56(2), 433–470.
- Wachter, J. (2010). Asset allocation. *Annual Reviews of Financial Economics* 2, 175–206.
- Welch, I. and A. Goyal (2008). A comprehensive look at the empirical performance of equity premium prediction. *The Review of Financial Studies* 21(4), 1455–1508.
- Xia, Y. (2001). Learning about predictability: the effects of parameter uncertainty on dynamic asset allocation. *The Journal of Finance* 56, 205–246.
- Zariphopoulou, T. (1999). Optimal investment and consumption models with non-linear stock dynamics. *Mathematical Methods for Operations Research* 50, 172–296.